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Abstract: The aim of this work is to present the analysis of the rate of convergence of a stochastic particle method for 1D viscous scalar conservation law equations. The convergence rate result is $\mathcal{O}(\Delta t + 1/\sqrt{N})$, where N is the number of numerical particles and Δt is the time step of the first order Euler scheme applied to the dynamic of the interacting particles.

Key-words: Stochastic particle method, stochastic processes, Euler scheme.

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Vitesse de convergence optimale d'une méthode particulaire stochastique appliquée à la résolution d'équations de conservation scalaire visqueuses unidimensionnelles

Résumé : Nous présentons l'analyse de la vitesse de convergence d'une méthode particulaire stochastique pour la résolution numérique d'équations de conservation scalaire visqueuse unidimensionnelles. Le résultat de vitesse de convergence est en $\mathcal{O}(\Delta t + 1/\sqrt{N})$, où N désigne le nombre de particules numériques et où Δt est le pas de discrétisation en temps du schéma d'Euler appliqué à l'équation de la dynamique des particules en interaction.

Mots-clés : Méthode particulaire stochastique, processus stochastiques, schéma d'Euler.

1 Introduction

We consider the following viscous scalar conservation law equation

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x) = \dfrac{\sigma^2}{2}\dfrac{\partial^2 V}{\partial x^2}(t,x) - \dfrac{\partial}{\partial x}A\left(V(t,x)\right), \ (t,x) \in (0,T] \times \mathbb{R}, \\ V(0,x) = V_0(x), \end{cases} \tag{1.1}$$

where $A : \mathbb{R} \to \mathbb{R}$ is a C^3 function. In this article we analyse the rate of convergence of a stochastic particle method for the numerical solution of Equation (1.1). When $A(v) = v^2/2$, Equation (1.1) is the viscous Burgers equation

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x) = \dfrac{\sigma^2}{2}\dfrac{\partial^2 V}{\partial x^2}(t,x) - V(t,x)\dfrac{\partial V}{\partial x}(t,x), \ (t,x) \in (0,T] \times \mathbb{R}, \\ V(0,x) = V_0(x). \end{cases} \tag{1.2}$$

A previous work proposes a stochastic particle method for the numerical solution of the Burgers equation (see Bossy and Talay [2, 3]). The method is based upon the probabilistic interpretation of the Burgers equation as the evolution equation of the cumulative distribution function of a stochastic nonlinear process (in the sense of McKean). The algorithm is inspired from a propagation of chaos result for the system of interacting particles associated to the nonlinear process. For the rate of convergence analysis, the initial condition of (1.2) is supposed to be the cumulative distribution function of a probability measure, or more generally, the distribution function of a signed and finite measure. Under suitable hypothesis on this initial (signed) measure, we proved a convergence rate of order $\mathcal{O}(1/\sqrt{N} + \sqrt{\Delta t})$ for the $L^1(\mathbb{R} \times \Omega)$ norm of the error, where N is the number of simulated interacting particles and Δt is the time step of the discretization by the Euler scheme of the stochastic differential system governing the particles motion.

For the Burgers case, numerical experiments confirm the order $\mathcal{O}(1/\sqrt{N})$ for the dependence on N, but show that the dependence in Δt is of order $\mathcal{O}(\Delta t)$ rather than $\mathcal{O}(\sqrt{\Delta t})$ (see [3, 1]). In this previous work, the error estimate was obtained by the use of the rate of convergence in $L^2(\Omega)$ of the Euler scheme whereas, in this sort of numerical computation, the averaging effect due to the propagation of chaos phenomena suggest to analyse the discretization error with estimates on the weak rate of convergence for the Euler scheme.

In this article, we extend our stochastic particle method for the Burgers equation in the general context of viscous scalar conservation law equation (1.1) and we prove a theoretical rate of convergence of order $\mathcal{O}(1/\sqrt{N} + \Delta t)$.

To construct our algorithm, we follow Jourdain [6] who gives a probabilistic interpretation of nonlinear parabolic PDE's such as the viscous scalar conservation law (1.1), when A is a C^1 function and V_0 is a non constant function with bounded variation. In [6], Jourdain provides a natural way to connect (1.1) with a nonlinear martingale problem and proves a propagation of chaos result for the suitable system of weakly interacting particles. Here we briefly present the main ideas and some results on which we base our numerical algorithm: we suppose that there exist a bounded and signed initial measure $m_0 \neq 0$ on $\mathbb{R}$ and a constant β such that $V_0(x) = \beta + H * m_0(x)$, where $H(x) = \mathbb{1}_{\{x \geq 0\}}$ denotes the Heaviside function. Let $|m_0|$, $\|m_0\|$ and h denote respectively the absolute value of m_0, the total variation of m_0 and a density of m_0 with respect to the probability measure $|m_0|/\|m_0\|$ with values in $\{-\|m_0\|, \|m_0\|\}$. For P a probability measure on $C([0,+\infty),\mathbb{R})$, we define the flow $(\tilde{P}_t)_{t\geq 0}$ of signed measures on $\mathbb{R}$ by

$$\forall B \in \mathcal{B}(\mathbb{R}), \quad \tilde{P}_t(B) = \mathbb{E}_P(\mathbb{1}_B(X_t)h(X_0)),$$

where X denotes the canonical process on $C([0,+\infty),\mathbb{R})$. We associate the following martingale problem to Equation (1.1):

Definition 1.1 *The probability measure $P \in \mathcal{P}(C([0,+\infty),\mathbb{R})$ is a solution of the nonlinear martingale problem $\mathcal{M}$, starting at m_0, if $P_0 = |m_0|/\|m_0\|$ and for each $\phi \in C_b^2(\mathbb{R})$,*

$$\phi(X_t) - \phi(X_0) - \int_0^t \left(\frac{\sigma^2}{2}\phi''(X_s) + A'(\beta + H * \tilde{P}_s(X_s))\phi'(X_s)\right) ds \text{ is a } P \text{ martingale.} \quad (1.3)$$

We define a system of N particles in mean field interaction by the following stochastic differential equation

$$\begin{aligned} X_t^{i,N} &= X_0^{i,N} + \sigma W_t^i + \int_0^t A'\left(\beta + \frac{1}{N}\sum_{j=1}^N H(X_s^{i,N} - X_s^{j,N})h(X_0^j)\right) ds, \\ &= X_0^{i,N} + \sigma W_t^i + \int_0^t A'\left(\beta + H * \tilde{\mu}_s^N(X_s^{i,N})\right) ds, \ 1 \leq i \leq N, \end{aligned}$$

where $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{X^{j,N}}$ is the empirical measure of the particles and $(W^1,\ldots,W^N)$ is a N-dimensional Brownian motion independent of the initial variables $(X_0^{1,N},\ldots,X_0^{N,N})$ which are I.I.D with law $|m_0|/\|m_0\|$.

Proposition 1.2 *(Jourdain [6]) The martingale problem $\mathcal{M}$ starting at m_0 admits a unique solution P and the particle systems $(X^{1,N},\ldots,X^{N,N})$ are P-chaotic. Moreover, Equation (1.1) has a unique bounded weak solution given by $V(t,x) = \beta + H * \tilde{P}_t(x)$.*

The propagation of chaos result implies that $\beta + \frac{1}{N}\sum_{i=1}^N H(x - X_t^{i,N})h(X_0^i) = \beta + H * \tilde{\mu}_t^N(x)$ converges in $L^1(\Omega)$ to $V(t,x)$ (see [6]). In practice the $X_t^{i,N}$'s cannot be computed exactly. The algorithm involves their approximation by a discrete-time stochastic process $(\bar{X}_{k\Delta t}^i, 1 \leq i \leq N)$, where Δt is a discretization step of the time interval $[0,T]$. Then we construct our approximation of $V(t,x)$ by

$$\bar{V}_{k\Delta t}^{N,\Delta t}(x) = \beta + \frac{1}{N}\sum_{i=1}^N w_i H(x - \bar{X}_{k\Delta t}^i),$$

where the weight w_i is determined by the initial position of the ith particle. Under smoothness hypotheses on V_0 and A, we prove that

$$\mathbb{E}\|V(T,\cdot) - \bar{V}_T^{N,\Delta t}(\cdot)\|_{L^1(\mathbb{R})} + \sup_{x\in\mathbb{R}} \left(\mathbb{E}\left|V(T,x) - \bar{V}_T^{N,\Delta t}(x)\right|\right) = \mathcal{O}\left(\frac{1}{\sqrt{N}} + \Delta t\right).$$

The first work on the optimal rate of convergence of the Euler scheme for interacting particle system is due to Kohatsu-Higa and Ogawa [7]. They analyse the convergence of the weak approximation of a general nonlinear diffusion process of the form

$$\begin{cases} dX_t = a(X_t, F * u_t(X_t))dt + b(X_t, G * u_t(X_t))dW_t, \text{where } u_t \text{ is the law of } X_t, \\ X_{t=0} = X_0 \text{ with law } m_0. \end{cases} \tag{1.4}$$

Suppose that the functions a, b, F and G are smooth with bounded derivatives, they use Malliavin calculus to show that, for any function f C^∞ whose derivatives have polynomial growth at infinity,

$$\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N} f(\bar{X}_{k\Delta t}^i) - \mathbb{E}f(X_{t_k})\right| \leq C(\frac{1}{\sqrt{N}} + \Delta t),$$

where C is independent of Δt and N but depends on f and $(\bar{X}_{k\Delta t}^i)_{i=1,\dots,N}$ is the corresponding discrete time system of interacting particles.

In the context on this present work, $a = A'$ but the diffusion coefficient b is a constant and F is the bounded but discontinuous Heaviside function H. Furthermore, we approximate the cumulative distribution function of X_t. The main difficulty of our rate of convergence analysis resides in the treatment of the kernel H. We do not use Malliavin calculus, but take part of the case we are of constant diffusion coefficient to adapt some techniques developed by Talay and Tubaro [9] in their study of the global error of the Euler scheme for stochastic differential equations linear in the sense of McKean. To do so, we strengthen the hypothesis in [2, 3] on the initial condition V_0 in order to work with the classical solution of Equation (1.1).

2 Algorithm and convergence rate

First, we state our hypotheses.

(H1) *The function A is of class C^3.*

(H2) *There exist a constant β and a signed and bounded measure on $\mathbb{R}$ $m_0 \neq 0$ such that the initial condition V_0 of Equation* (1.1) *is given by*

$$V_0(x) = \beta + H * m_0(x).$$

(H3) *The measure m_0 is absolutely continuous with respect to the Lesbegue measure. Its density U_0 is a bounded function with bounded first order derivative. Moreover, there exist strictly positive constants M, η and α such that $|U_0|(x) \leq \eta \exp(-\alpha \frac{x^2}{2})$, for all $|x| > M$.*

We construct a family $(y_0^i, w^i)_{1\leq i\leq N}$ of weighted initial particles such that the piecewise constant function

$$\overline{V}_0(x) = \beta + \frac{1}{N}\sum_{i=1}^{N} w^i H(x - y_0^i)$$

approximates $V_0(x)$. Many strategies are possible. As suggested in [6], one can choose the density

$$h(x) = \|m_0\| \left(1\!\!1_{\{U_0(x)\geq 0\}} + 1\!\!1_{\{U_0(x)<0\}}\right).$$

The N locations (y_0^i) are chosen in order to approximate the distribution function of the measure $|m_0|/\|m_0\|$ by a piecewise constant function. We invert the distribution function

$$y_0^i = \begin{cases} \inf\{y; \int_{-\infty}^{y} |m_0|/\|m_0\|(dx) = \frac{i}{N}\},\ i = 1, \ldots, N-1, \\ \inf\{y; \int_{-\infty}^{y} |m_0|/\|m_0\|(dx) = 1 - \frac{1}{2N}\},\ i = N \end{cases}$$

and the corresponding weights are $w_i = h(y_0^i)$, for $i = 1, \ldots, N$. Let $\overline{m}_0$ denote the associated empirical measure

$$\overline{m}_0 = \frac{1}{N}\sum_{i=1}^{N} w^i \delta_{y_0^i}. \tag{2.1}$$

By construction,

$$\|V_0 - \overline{V}_0\|_{L^\infty(\mathbb{R})} \leq \|m_0\|/N. \tag{2.2}$$

The convergence for the $L^1(\mathbb{R})$ norm is described by

Lemma 2.1 *(Bossy&Talay [3]). Assume (H3). There exists a constant C depending on m_0 only such that*

$$\|V_0 - \overline{V}_0\|_{L^1(\mathbb{R})} \leq C\sqrt{\log(N)}/N.$$

If the density U_0 has a compact support, the bound is C/N.

On a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, we consider a N-dimensional $(\mathcal{F}_t)$-Brownian motion $(W^1, \ldots, W^N)$. As suggested by the propagation of chaos result in Proposition 1.2, to construct an approximation of $V(t,x)$, we have to move our N particles according to the following system of stochastic differential equations

$$\begin{cases} dX_t^i = \sigma dW_t^i + A'\left(\beta + \frac{1}{N}\sum_{j=1}^{N} w^j H(X_t^i - X_t^j)\right) dt, \ i = 1,\dots,N, \\ X_0^i = y_0^i. \end{cases}$$

The piecewise constant function

$$\widehat{V}(t,x) = \beta + \frac{1}{N}\sum_{i=1}^{N} w^i H(x - X_t^i)$$

approximates $V(t,x)$ with an error depending on N only. To get a simulation procedure for a trajectory of each (X^i), we discretize in time. Chose Δt and $K \in \mathbb{N}$ such that $T = \Delta t K$ and denote by $t_k = k\Delta t$ the discrete times, with $1 \le k \le K$. The Euler scheme leads to the following discrete-time system

$$\begin{cases} Y_{t_{k+1}}^i = Y_{t_k}^i + \sigma\left(W_{t_{k+1}}^i - W_{t_k}^i\right) + \Delta t A'\left(\beta + \frac{1}{N}\sum_{j=1}^{N} w^j H(Y_{t_k}^i - Y_{t_k}^j)\right), \ i = 1,\dots,N, \\ Y_0^i = y_0^i \end{cases}$$

Thus we approximate $V(t_k, x)$, solution of (1.1), by the piecewise constant function

$$\overline{V}_{t_k}(x) = \beta + \frac{1}{N}\sum_{i=1}^{N} w^i H(x - Y_{t_k}^i). \tag{2.3}$$

Our estimate on the convergence rate is

Theorem 2.2 *Assume (H1), (H2) and (H3). For $T > 0$ fixed, let $\Delta t > 0$ be such that $T = \Delta t K$, $K \in \mathbb{N}$. Let $V(t_k, x)$ be the solution at time $t_k = k\Delta t$ of Equation* (1.1) *with initial condition V_0. Let $\overline{V}_{t_k}(x)$ be defined as in* (2.3) *with N particles. Then there exists a strictly positive constant C, depending on V_0, A, σ and T only such that for all k in $\{1,\dots,K\}$,*

$$\sup_{x\in\mathbb{R}} \mathbb{E}\left|V(t_k,x) - \overline{V}_{t_k}(x)\right| \le C\left(\|V_0 - \overline{V}_0\|_{L^\infty(\mathbb{R})} + \frac{1}{\sqrt{N}} + \Delta t\right)$$

and

$$\mathbb{E}\left\|V(t_k,\cdot) - \overline{V}_{t_k}(\cdot)\right\|_{L^1(\mathbb{R})} \le C\left(\|V_0 - \overline{V}_0\|_{L^1(\mathbb{R})} + \frac{1}{\sqrt{N}} + \Delta t\right).$$

3 Proof of Theorem 2.2

Without lost of generality, we simplify the presentation of the proof by restricting ourself to the case where, in assumption (H2), the constant $\beta = 0$ and V_0 is the distribution function of a probability measure. Thus, we replace the assumption (H2) by

(H2′) *There exists a probability measure m_0 such that the initial condition V_0 of Equation* (1.1) *is given by*

$$V_0(x) = H * m_0(x).$$

The weights w^i for $i = 1, \ldots, N$ are now all equal to 1. Thus, we approximate $V(t_k, x)$, solution of (1.1) at time t_k, by the piecewise constant function

$$\overline{V}_{t_k}(x) = \frac{1}{N} \sum_{i=1}^{N} H(x - Y^i_{t_k}),$$

with

$$\begin{cases} Y^i_{t_{k+1}} = Y^i_{t_k} + \sigma \left(W^i_{t_{k+1}} - W^i_{t_k} \right) + \Delta t A' \left(\frac{1}{N} \sum_{j=1}^{N} H(Y^i_{t_k} - Y^j_{t_k}) \right), i = 1, \ldots, N, \\ Y^i_0 = y^i_0. \end{cases}$$

In the sequel, we will use the continuous version of discrete time processes (Y^i) which consists in freezing the drift coefficient on each interval $[t_k, t_{k+1}]$:

$$Y^i_t = y^i_0 + \int_0^t A' \left(\overline{V}_{\eta(s)}(Y^i_{\eta(s)}) \right) ds + \sigma W^i_t, \tag{3.1}$$

where $\eta(s) = \sup_{k \in [0, \ldots, K]} \{t_k; t_k \leq s\}$. C denotes any positive constant depending only on T, σ, A and V_0; for any positive constant α, g_α denotes the Gaussian density function

$$g_\alpha(x) = \frac{1}{\sqrt{2\pi\alpha}} \exp\left(-\frac{x^2}{2\alpha} \right).$$

According to the probabilistic interpretation given in Section 1 under (H2′), the solution of Equation (1.1) is given by

$$V(t, x) = \mathbb{E}_P(\mathbb{1}_{(-\infty, x]}(X_t)),$$

where P is the solution of the martingale problem 1.3 and X denotes the canonical process on $C([0, +\infty), \mathbb{R})$. We define the real valued function $B(t, x)$ by

$$B(t, x) = A' \left(\mathbb{E}_P(\mathbb{1}_{(-\infty, x]}(X_t)) \right) = A'(V(t, x)), \ (t, x) \in [0, T] \times \mathbb{R}. \tag{3.2}$$

Now consider the Markov process (Z) solution of

$$\begin{cases} dZ_t = B(t, Z_t)dt + \sigma dW_t, \\ Z_{t=0} = Z_0 \text{ with law } m_0, \end{cases} \tag{3.3}$$

where (W) is a one-dimensional Brownian motion independent of $(W^1, \dots, W^N)$. Proposition 1.2 provides the existence in law of (Z) and thus $V(t,x) = \mathbb{E}H(x - Z_t)$.

Let $k \in \{1, \dots, K\}$. To prove the Theorem 2.2, we start from

$$V(t_k, x) - \overline{V}_{t_k}(x) = \mathbb{E}H(x - Z_{t_k}) - \frac{1}{N}\sum_{i=1}^{N} H(x - Y^i_{t_k}).$$

First, we introduce an artificial smoothing of the Heaviside function. For an arbitrary constant $\varepsilon > 0$, we define the function $H_\varepsilon(x) = g_\varepsilon * H(x)$ and we decompose the expression above into four parts:

$$\begin{aligned} V(t_k, x) - \overline{V}_{t_k}(x) \quad &= \mathbb{E}H(x - Z_{t_k}) - \mathbb{E}H_\varepsilon(x - Z_{t_k}) \\ &+ \frac{1}{N}\sum_{i=1}^{N}\left[H_\varepsilon(x - Y^i_{t_k}) - H(x - Y^i_{t_k})\right] \\ &+ \int_{\mathbb{R}} \mathbb{E}H_\varepsilon(x - Z^{y,0}_{t_k}) m_0(dy) - \int_{\mathbb{R}} \mathbb{E}H_\varepsilon(x - Z^{y,0}_{t_k})\overline{m}_0(dy) \\ &+ \frac{1}{N}\sum_{i=1}^{N}\left[\mathbb{E}H_\varepsilon(x - Z^{y^i_0,0}_{t_k}) - H_\varepsilon(x - Y^i_{t_k})\right], \end{aligned} \tag{3.4}$$

where $(Z^{y,0})$ is the solution of the stochastic differential equation (3.3) starting at y at time 0. The two first terms are smoothing errors and will be bound by a constant arbitrary small depending on ε. The third term corresponds to the propagation at time t_k of the initialization error $|V_0(x) - \overline{V}_0(x)|$. To make understand the last term, we transform it. For any time t_k and any $x \in \mathbb{R}$, we consider the partial differential equation

$$\begin{cases} \dfrac{\partial v_{t_k,x}}{\partial s}(s,y) + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 v_{t_k,x}}{\partial y^2}(s,y) + B(s,y)\dfrac{\partial v_{t_k,x}}{\partial y}(s,y) = 0, \ \forall (s,y) \in [0, t_k) \times \mathbb{R}, \\ v_{t_k,x}(t_k, y) = H_\varepsilon(x - y), \ \forall y \in \mathbb{R}. \end{cases} \tag{3.5}$$

From the Lemma 3.9 below, $v_{t_k,x}$ is a bounded function in $C^{1,2}((0,t_k] \times \mathbb{R})$ and by the the Feynman-Kac representation of a Cauchy problem, $v_{t_k,x}(s,y) = \mathbb{E}H_\varepsilon(x - Z^{s,y}_{t_k})$. Hence,

$$\mathbb{E}H_\varepsilon(x - Z^{y^i_0,0}_{t_k}) - H_\varepsilon(x - Y^i_{t_k}) \quad = \quad v_{t_k,x}(0, y^i_0) - v_{t_k,x}(t_k, Y^i_{t_k}).$$

As $v_{t_k,x}$ is solution of (3.5), Itô's formula gives

$$
\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\left[\mathbb{E}H_\varepsilon(x - Z_{t_k}^{y_0^i,0}) - H_\varepsilon(x - Y_{t_k}^i)\right] \\
&= \frac{1}{N}\sum_{i=1}^{N}\int_0^{t_k}\frac{\partial v_{t_k,x}}{\partial y}(s, Y_s^i)\left[B(s, Y_s^i) - A'\left(\overline{V}_{\eta(s)}(Y_{\eta(s)}^i)\right)\right]ds \\
&- \frac{1}{N}\sum_{i=1}^{N}\int_0^{t_k}\sigma\frac{\partial v_{t_k,x}}{\partial y}(s, Y_s^i)dW_s^i.
\end{aligned}
\tag{3.6}
$$

The second term of the right-hand side is a statistical error. We will bound the expectation of its absolute value by $C/\sqrt{N}$. The first term in the right-hand side of (3.6) is the discretization error and concentrates the most important difficulties of the proof.

In the next subsections, we give the proof of the following four lemmas:

Lemma 3.1 *Smoothing error.*
For any $z \in \mathbb{R}$, $\int_{\mathbb{R}} |H(x-z) - H_\varepsilon(x-z)|dx \le C\sqrt{\varepsilon}$. *Under (H1), (H2′) and (H3),*

$$\sup_{x\in\mathbb{R}} |\mathbb{E}H(x - Z_{t_k}) - \mathbb{E}H_\varepsilon(x - Z_{t_k})| \le C\sqrt{\varepsilon} \tag{3.7}$$

and for any i *and* j *in* $\{1,\ldots,N\}$ *and any* k *in* $\{1,\ldots,K\}$,

$$
\begin{aligned}
&\sup_{x\in\mathbb{R}} \mathbb{E}\left|H_\varepsilon(x - Y_{t_k}^i) - H(x - Y_{t_k}^i)\right| \le C\sqrt{\varepsilon}/\sqrt{\Delta t}, \\
&\mathbb{E}\left|H_\varepsilon(Y_{t_k}^j - Y_{t_k}^i) - H(Y_{t_k}^i - Y_{t_k}^i)\right| \le C\sqrt{\varepsilon}/\sqrt{\Delta t}.
\end{aligned}
\tag{3.8}
$$

The positive constant C *depends on* V_0, A, σ *and* T *only.*

Lemma 3.2 *Initialization error.*
Assume (H1), (H2′) and (H3). For all $k \in \{1,\ldots,K\}$,

$$\sup_{x\in\mathbb{R}}\left|\int_{\mathbb{R}}\mathbb{E}H_\varepsilon(x - Z_{t_k}^{y,0})m_0(dy) - \int_{\mathbb{R}}\mathbb{E}H_\varepsilon(x - Z_{t_k}^{y,0})\overline{m}_0(dy)\right| \le C\|V_0 - \overline{V}_0\|_{L^\infty(\mathbb{R})}$$

and

$$\left\|\int_{\mathbb{R}}\mathbb{E}H_\varepsilon(x - Z_{t_k}^{y,0})m_0(dy) - \int_{\mathbb{R}}\mathbb{E}H_\varepsilon(x - Z_{t_k}^{y,0})\overline{m}_0(dy)\right\|_{L^1(\mathbb{R})} \le C\|V_0 - \overline{V}_0\|_{L^1(\mathbb{R})}.$$

The positive constant C *depends on* V_0, A, σ *and* T *only.*

Lemma 3.3 *Statistical error.*
Assume (H1), (H2′) and (H3). For all $k \in \{1,\ldots,K\}$,

$$\sup_{x\in\mathbb{R}}\mathbb{E}\left|\sum_{i=1}^{N}\frac{w^i}{N}\int_0^{t_k}\sigma\frac{\partial v_{t_k,x}}{\partial y}(s, Y_s^i)dW_s^i\right| \le \frac{C}{\sqrt{N}}$$

and

$$\mathbb{E}\left\| \sum_{i=1}^{N} \frac{w^i}{N} \int_0^{t_k} \sigma \frac{\partial v_{t_k,\cdot}}{\partial y}(s, Y_s^i) dW_s^i \right\|_{L^1(\mathbb{R})} \leq \frac{C}{\sqrt{N}}.$$

The positive constant C depends on V_0, A, σ and T only.

Lemma 3.4 *Discretization error.*
Assume (H1), (H2′) and (H3). For all $k \in \{1, \ldots, K\}$,

$$\sup_{x\in\mathbb{R}} \mathbb{E}\left| \frac{1}{N} \sum_{i=1}^{N} \int_0^{t_k} \frac{\partial v_{t_k,x}}{\partial y}(s, Y_s^i) \left(B(s, Y_s^i) - A'\left(\overline{V}_{\eta(s)}(Y_{\eta(s)}^i) \right) \right) ds \right| \leq C\left(\frac{1}{\sqrt{N}} + \Delta t\right)$$

and

$$\mathbb{E}\left\| \frac{1}{N} \sum_{i=1}^{N} \int_0^{t_k} \frac{\partial v_{t_k,x}}{\partial y}(s, Y_s^i) \left(B(s, Y_s^i) - A'\left(\overline{V}_{\eta(s)}(Y_{\eta(s)}^i) \right) \right) ds \right\|_{L^1(\mathbb{R})} \leq C\left(\frac{1}{\sqrt{N}} + \Delta t\right).$$

The positive constant C depends on V_0, A, σ and T only.

Chose $\varepsilon = \Delta t^3$. Estimates of the four above lemmas combined with Equalities (3.6) and (3.4) lead to the Theorem 2.2.

The section is organised as follows. In Subsection 3.1, we prove some preliminary estimates and regularity results on the drift function B and on the solution $v_{t_k,x}$ of Equation (3.5). Then in the successive subsections, we prove Lemmas 3.1, 3.2 and 3.3. We finish by the proof of the main Lemma 3.4.

3.1 Preliminary lemmas

Consider the process (Z) solution of (3.3). The drift function B defined in (3.2) is bounded by $\sup_{[0,1]} |A'(v)|$. Hence, by Girsanov's theorem, for any $t > 0$, Z_t has a density denoted by $U(t, \cdot)$. Furthermore,

Remark 3.5 *The transition probability $\mathbb{P}(t, dz; s, Z_s = y)$ has a density, that we denote by $\Gamma(t, z; s, y)$, which is in $L^2(\mathbb{R})$. Moreover for all $y \in \mathbb{R}$,*

$$\|\Gamma(t, \cdot; s, y)\|_{L^2(\mathbb{R})} \leq \frac{C}{(t-s)^{1/4}},$$

where the positive constant C depends on σ, T and A only and, therefore, is uniform in y. This can be proven by using Girsanov's theorem (see the proof of Proposition 1.1 in [8]). In particular, $U(t, \cdot)$ is in $L^2(\mathbb{R})$ for all $t > 0$ and, without any hypothesis on m_0,

$$\|U(t, \cdot)\|_{L^2(\mathbb{R})} \leq \frac{C}{t^{1/4}}.$$

Lemma 3.6 *Assume (H1), (H2') and* $(H3)$. *The density* $U(t,x)$ *of* Z_t *is bounded uniformly in* $t \in [0,T]$ *and has a first partial derivative in* x *which is bounded uniformly in* $t \in [0,T]$. *The function* $B(t,x)$ *is in* $C_b^{1,2}([0,T]\times\mathbb{R})$ *and its derivatives* $\frac{\partial B}{\partial t}(t,x)$, $\frac{\partial B}{\partial x}(t,x)$ *and* $\frac{\partial^2 B}{\partial x^2}(t,x)$ *are bounded uniformly in* $t \in [0,T]$.

Remark 3.7 *Even if this is not explicitly stated in Lemma 3.6, one can easily deduce from the following proof that* V *is in* $C_b^{1,2}([0,T]\times\mathbb{R})$ *and thus is the classical solution of the scalar conservation law Equation* (1.1).

Proof : For all $t > 0$, $g_{\sigma^2 t}(x)$ denotes the density of the Gaussian random variable σW_t. Let S_t be the corresponding semi-group defined by $S_t f = g_{\sigma^2 t} * f$. Let us show that U is the unique weak solution in $L^1(\mathbb{R})$ of the following integral linear Fokker Planck equation

$$p_t = S_t U_0 - \int_0^t \frac{\partial}{\partial x} S_{t-s}\left(B(s,\cdot)p_s\right) ds, \ \forall\, t \in]0,T], \ \ p_0 = U_0, \tag{3.9}$$

where U_0 is the density of m_0. We will deduce from Equation (3.9) the regularity results of the Lemma. For a fixed t in $(0,T]$ and a function f in $C^\infty(\mathbb{R})$ with compact support, we set $G(s,x) = S_{t-s}f(x)$, for all $s \in [0,t)$. Then G is the solution of the backward heat equation

$$\begin{cases} \dfrac{\partial G}{\partial s} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 G}{\partial x^2} = 0, \quad 0 \le s < t, \\ G(t,x) = f(x). \end{cases}$$

By applying Itô's formula to $G(t,Z_t)$ and taking the expectation, we obtain that

$$\mathbb{E}G(t,Z_t) = \int_{\mathbb{R}} f(x)U(t,x)dx = \int_{\mathbb{R}} G(0,x)U_0(x)dx + \int_0^t \int_{\mathbb{R}} \frac{\partial G}{\partial x}(s,x)B(s,x)U(s,x)dx.$$

Successive substitutions in the second term of the right-hand side lead to

$$\int_{\mathbb{R}} f(x)U(t,x)dx = \int_{\mathbb{R}} S_t f(x)U_0(x)dx + \int_0^t \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g'_{\sigma^2(t-s)}(x-y)f(y)dy\right) B(s,x)U(s,x)dxds$$

and

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g'_{\sigma^2(t-s)}(x-y)f(y)dy\right) B(s,x)U(s,x)dxds \\ &= \int_0^t \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} g'_{\sigma^2(t-s)}(x-y)B(s,x)U(s,x)dx\right) dyds \\ &= -\int_0^t \int_{\mathbb{R}} f(y)\frac{\partial}{\partial y} \left(\int_{\mathbb{R}} g_{\sigma^2(t-s)}(x-y)B(s,x)U(s,x)dx\right) dyds \\ &= -\int_0^t \int_{\mathbb{R}} f(y)\frac{\partial}{\partial y} S_{t-s}\left(B(s,\cdot)U(s,\cdot)\right)(y)dyds. \end{aligned}$$

Hence, we obtain that

$$\int_{\mathbb{R}} f(x)U(t,x)dx = \int_{\mathbb{R}} f(x)S_t U_0(x)dx - \int_0^t \int_{\mathbb{R}} f(x)\frac{\partial}{\partial x} S_{t-s}\left(B(s,\cdot)U(s,\cdot)\right)(x)dxds$$

which means that U satisfies Equation (3.9) in the weak sense. Consider now two solutions p^1 and p^2 in $L^1(\mathbb{R})$ of Equation (3.9). Then, for all $t \in (0,T]$,

$$\begin{aligned}
\|p_t^1 - p_t^2\|_{L^1(\mathbb{R})} &= \left\| \int_0^t \frac{\partial}{\partial x} S_{t-s}\left(B(s,\cdot)(p_s^1 - p_s^2)\right) ds \right\|_{L^1(\mathbb{R})} \\
&\leq \sup_{u\in[0,1]} |A(u)'| \int_0^t \left\| g'_{\sigma^2(t-s)} \right\|_{L^1(\mathbb{R})} \left\| p_s^1 - p_s^2 \right\|_{L^1(\mathbb{R})} ds \\
&\leq \int_0^t \frac{C}{\sqrt{t-s}} \left\| p_s^1 - p_s^2 \right\|_{L^1(\mathbb{R})} ds.
\end{aligned}$$

We conclude on the uniqueness of the solution of (3.9) by applying Gronwall's lemma. We have now that for all $t \in (0,T]$ and $x \in \mathbb{R}$,

$$U(t,x) = g_{\sigma^2 t} * U_0(x) - \int_0^t g'_{\sigma^2(t-s)} * \left(B(s,\cdot)U(s,\cdot)\right)(x)ds. \tag{3.10}$$

Let us prove that U is bounded uniformly in $t \in [0,T]$. By Remark 3.5,

$$\begin{aligned}
U(t,x) &\leq \|U_0\|_{L^\infty(\mathbb{R})} + \sup_{[0,1]} |A'| \int_0^t \int_{\mathbb{R}} |g'_{\sigma^2(t-s)}|(x-y)U(s,y)dyds \\
&\leq \|U_0\|_{L^\infty(\mathbb{R})} + \int_0^t \frac{C}{\sqrt{(t-s)}} \sqrt{\int_{\mathbb{R}} g_{2\sigma^2(t-s)}(x-y)U^2(s,y)dy} ds \\
&\leq \|U_0\|_{L^\infty(\mathbb{R})} + \int_0^t \frac{C}{(t-s)^{3/4}s^{1/4}} ds.
\end{aligned}$$

Thus $\|U\|_{L^\infty([0,T]\times\mathbb{R})} \leq C$, where the constant C depends on σ, T, A and U_0 only. Now we remark that $\frac{\partial B}{\partial x}(t,\cdot) = A''\left(\mathbb{E}H(x-Z_t)\right)U(t,\cdot)$ in $L^\infty([0,T]\times\mathbb{R})$ and hence that $\left\|\frac{\partial B}{\partial x}\right\|_{L^\infty([0,T]\times\mathbb{R})} \leq C$.

If we formally derive (3.10), we obtain that $\frac{\partial U}{\partial x}$ must satisfy the equation

$$\frac{\partial U}{\partial x}(t,x) = g_{\sigma^2 t} * U_0'(x) - \int_0^t g'_{\sigma^2(t-s)} * \left(\frac{\partial B}{\partial x}(s,\cdot)U(s,\cdot) + B(s.\cdot)\frac{\partial U}{\partial x}(s,\cdot)\right)(x)ds. \tag{3.11}$$

Let us prove that $\frac{\partial U}{\partial x}$ satisfies (3.11) and more precisely that $\frac{\partial U}{\partial x}$ is in $C([0,T], L^1(\mathbb{R}) \cap C_b(\mathbb{R}))$. Let $E_{[0,T]}$ be the space

$$E_{[0,T]} = \left\{ u \in C([0,T], L^1(\mathbb{R}) \cap C^b(\mathbb{R})), \|u\|_{E_{[0,T]}} = \sup_{t\in[0,T]} \|u(t)\|_E < +\infty \right\},$$

where $\|f\|_E = \|f\|_{L^1(\mathbb{R})} + \sup_{x\in\mathbb{R}} |f(x)| + \| \int_{-\infty}^{\cdot} f(y)dy\|_{L^1(\mathbb{R})}$. Let $\Upsilon : E_{[0,T]} \longrightarrow E_{[0,T]}$ be defined by

$$\Upsilon(u)(t,x) = g_{\sigma^2 t} * U_0'(x) - \int_0^t g'_{\sigma^2(t-s)} * \left(\frac{\partial B}{\partial x}(s,\cdot) \left(\int_{-\infty}^{\cdot} u(s,y)dy \right) + B(s,\cdot)u(s,\cdot) \right)(x)ds.$$

We show that $\frac{\partial U}{\partial x}$ is the fix point in $E_{[0,T]}$ of the application Υ. For u^1 and u^2 in $E_{[0,T]}$,

$$\begin{aligned} &\left(\Upsilon(u^1) - \Upsilon(u^2)\right)(t,x) \\ &= \int_0^t g'_{\sigma^2(t-s)} * \left(\frac{\partial B}{\partial x}(s,\cdot) \left(\int_{-\infty}^{\cdot} (u^1-u^2)(s,y)dy \right) + B(s,\cdot)(u^1-u^2)(s,\cdot) \right)(x)ds. \end{aligned}$$

An easy computation shows that

$$\left\| (\Upsilon(u^1) - \Upsilon(u^2))(t) \right\|_E \leq \int_0^t \|g'_{\sigma^2(t-s)}\|_{L^1(\mathbb{R})} \left\| |B| + |\frac{\partial B}{\partial x}| \right\|_{L^\infty([0,T]\times\mathbb{R})} \|(u^1-u^2)(s)\|_E \; ds.$$

Let t_0 such that $\displaystyle\int_0^{t_0} \frac{2D}{\sqrt{2\pi\sigma^2(t-s)}}ds = \frac{1}{2}$, where $D = \left\| |B| + |\frac{\partial B}{\partial x}| \right\|_{L^\infty([0,T]\times\mathbb{R})}$. Then we deduce from the previous inequality that Υ is a contraction on $E_{[0,t_0]}$ and we note ν its fix point. For any u in $E_{[0,T]}$ and $t \in (t_0, T]$, we remark that

$$\begin{aligned} \Upsilon(u)(t,x) = &g_{\sigma^2(t-t_0)} * \Upsilon(u(t_0))(x) \\ &- \int_{t_0}^t g'_{\sigma^2(t-s)} * \left(\frac{\partial B}{\partial x}(s,\cdot) \left(\int_{-\infty}^{\cdot} u(s,y)dy \right) + B(s,\cdot)u(s,\cdot) \right)(x)ds. \end{aligned}$$

Now if ν^1 and ν^2 in $E_{[0,2t_0]}$ are such that $\nu^1(t) = \nu^2(t) = \nu(t)$ for $t \in [0,t_0]$, from the equality above we easily get that

$$\left\| (\Upsilon(\nu^1) - \Upsilon(\nu^2))(t) \right\|_E \leq \int_{t_0}^{2t_0} \|g'_{\sigma^2(t-s)}\|_{L^1(\mathbb{R})} \left\| |B|(s,\cdot) + |\frac{\partial B}{\partial x}(s,\cdot)| \right\|_{L^\infty(\mathbb{R})} \|(\nu^1-\nu^2)(s)\|_E ds$$

and then $\left\| (\Upsilon(\nu^1) - \Upsilon(\nu^2))(t) \right\|_{E_{[t_0,2t_0]}} \leq \frac{1}{2} \|(\nu^1-\nu^2)\|_{E_{[t_0,2t_0]}}$. Repeating this procedure, we construct the fix point ν of Υ on $E_{[0,T]}$. Finally, we remark that the function $(t,x) \longrightarrow \int_{-\infty}^x \nu(t,y)dy$ is solution in $L^1(\mathbb{R})$ of Equation (3.9). Then by the uniqueness of the solution of (3.9), $U(t,x) = \int_{-\infty}^x \nu(t,y)dy$ and $\frac{\partial U}{\partial x}(t,x) = \nu(t,x)$. Moreover from (3.11),

$$\|\frac{\partial U}{\partial x}(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \|U_0'\|_{L^\infty(\mathbb{R})} + \frac{\sqrt{T}D}{\sqrt{2\pi\sigma^2}} \|U\|_{L^\infty([0,T]\times\mathbb{R})} + \int_0^t \frac{2D}{\sqrt{2\pi\sigma^2(t-s)}} \|\frac{\partial U}{\partial x}(s,\cdot)\|_{L^\infty(\mathbb{R})} ds.$$

Thus by Gronwall's lemma, we conclude that $\frac{\partial U}{\partial x}$ is bounded uniformly in $t \in [0,T]$. Moreover for all $(t,x) \in [0,T] \times \mathbb{R}$, we have

$$\frac{\partial B}{\partial x}(t,x) = A''\left(\mathbb{E}H(x - z_t)\right) U(t,x),$$
$$\frac{\partial^2 B}{\partial x^2}(t,x) = A'''\left(\mathbb{E}H(x - z_t)\right) U^2(t,x) + A''\left(\mathbb{E}H(x - z_t)\right) \frac{\partial U}{\partial x}(t,x)$$

and $\|\frac{\partial^2 B}{\partial x^2}\|_{L^\infty([0,T]\times\mathbb{R})} \leq C$. To finish the proof, we have now to bound the derivative in time of the function B. We have

$$\frac{\partial}{\partial t} B(t,x) = A''(\mathbb{E}H(x - Z_t)) \frac{\partial}{\partial t} \mathbb{E}H(x - Z_t),$$

where by (3.10),

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbb{E}H(x - Z_t) &= \frac{\partial}{\partial t}\int_{-\infty}^{x} U(t,y)dy \\
&= \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{x} g_{\sigma^2 t} * U_0(y)dy - \frac{\partial}{\partial t}\int_0^t g_{\sigma^2(t-s)} * \left(B(s,\cdot)U(s,\cdot)\right)(x)ds \\
&= \frac{\sigma^2}{2} g_{\sigma^2 t} * U_0' - B(t,x)U(t,x) - \int_0^t \frac{\sigma^2}{2} g'_{\sigma^2(t-s)} * \frac{\partial}{\partial x}\left(B(s,\cdot)U(s,\cdot)\right)(x)ds \\
&\leq \|U_0'\|_{L^\infty(\mathbb{R})} + \|BU\|_{L^\infty([0,T]\times\mathbb{R})} + \int_0^t \frac{C}{\sqrt{t-s}} ds,
\end{aligned}$$

which gives that $\left\| \frac{\partial}{\partial t} B(t,x)\right\|_{L^\infty([0,T]\times\mathbb{R})} \leq C$. ∎

The following lemma, due to Friedman, gives exponential bound for the transition density of $Z_t^{s,x}$.

Lemma 3.8 (Friedman [5], p.139-150, [4], chap.1)
If the drift function $B(t,x)$ is bounded on $[0,T]\times\mathbb{R}$, Lipschitz in x uniformly in t and Holder in t uniformly in x then the transition probability of the process $(Z_t^{x,s})$ has a smooth density, denoted by $\Gamma(t,z;s,x)$, and for any $T > 0$, there exists a constant C_0 depending on T, B and σ, such that for all $0 \leq s \leq t \leq T$ and $(x,z) \in \mathbb{R}^2$,

$$\Gamma(t,z;s,x) \ \leq \ \frac{C_0}{\sqrt{t-s}} \ \exp\left(-\frac{(x-z)^2}{4\sigma^2(t-s)}\right).$$

Lemma 3.9 *Assume (H1), (H2′) and (H3). The Cauchy problem* (3.5) *has a unique bounded solution in $C^{1,2}((0,t_k] \times \mathbb{R})$ and there exists a constant C depending only on A, σ, T and V_0 such that for all $(s,z) \in (0,t_k] \times \mathbb{R}$,*

$$\left|\frac{\partial v_{t_k,x}}{\partial z}(s,z)\right| \leq C g_{\varepsilon+2\sigma^2(t_k-s)}(x-z). \tag{3.12}$$

Moreover, for all $s \in [0, t_k)$

$$\sup_{z\in\mathbb{R}} \left\| \frac{\partial^2 v_{t_k,\cdot}}{\partial z^2}(s,z) \right\|_{L^1(\mathbb{R})} \leq \frac{C}{\sqrt{t_k - s}} \tag{3.13}$$

and

$$\sup_{x\in\mathbb{R}} \left\| \left| \frac{\partial^2 v_{t_k,x}}{\partial y^2} \right|^{5/4} (s,\cdot) \right\|_{L^1(\mathbb{R})} \leq \frac{C}{(t_k - s)^{3/4}}. \tag{3.14}$$

Proof : Existence and uniqueness of a bounded solution of Equation (3.5) can be found in Friedman [5]. Moreover, the Feynman-Kac representation holds and $v_{t_k,x}(s,y) = \mathbb{E}H_\varepsilon(x - Z_{t_k}^{s,y})$. Then we have

$$\frac{\partial v_{t_k,x}}{\partial y}(s,y) \quad = \quad \mathbb{E}\left[g_\varepsilon(x - Z_{t_k}^{s,y}) \frac{dZ_{t_k}^{s,y}}{dy} \right],$$

where

$$\frac{dZ_t^{0,y}}{dy} = \exp\left(\int_0^t \frac{\partial B}{\partial x}(s, Z_s^{0,y}) ds \right).$$

As the function $\frac{\partial B}{\partial x}(t,x)$ is bounded in $[0,T] \times \mathbb{R}$, we get

$$\left| \frac{\partial v_{t_k,x}}{\partial y}(s,y) \right| \leq C g_\varepsilon * \Gamma(t_k, \cdot; s, y)(x)$$

from which we deduce immediately (3.12), by Lemma 3.8.

For all $(s,y) \in [0,t_k) \times \mathbb{R}$, we define the function $u_{t_k,x}(s,y) = v_{t_k,x}(t_k - s, y)$. Then $u_{t_k,x}(s,z)$ is the unique strong solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial u_{t_k,x}}{\partial s}(s,y) = \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 u_{t_k,x}}{\partial y^2}(s,y) + B(t_k - s, y) \dfrac{\partial u_{t_k,x}}{\partial y}(s,y) = 0, \ \forall (s,y) \in [0,t_k) \times \mathbb{R}, \\ u_{t_k,x}(0,y) = H_\varepsilon(x-y), \ \forall x \in \mathbb{R}. \end{cases}$$

We easily deduce from this equation that for all $(s,y) \in [0,t_k) \times \mathbb{R}$,

$$\begin{aligned} u_{t_k,x}(s,y) &= S_s u_{t_k,x}(0,\cdot)(y) + \int_0^s S_{s-\theta} \left(B(t-\theta,\cdot) \frac{\partial u_{t_k,x}}{\partial y}(\theta,\cdot) \right)(y) d\theta \\ &= \int_{-\infty}^y g_{\sigma^2 s + \varepsilon}(z-x) dz + \int_0^s \int_{\mathbb{R}} g_{\sigma^2(s-\theta)}(y-z) B(t-\theta,z) \frac{\partial u_{t_k,x}}{\partial y}(\theta,z) dz d\theta. \end{aligned}$$

Deriving two times the expression below, we get

$$\frac{\partial^2 u_{t_k,x}}{\partial y^2}(s,y) = g'_{\sigma^2 s+\varepsilon}(y-x) + \int_0^s \int_{\mathbb{R}} g'_{\sigma^2(s-\theta)}(y-z) \frac{\partial}{\partial z} \left(B(t-\theta,z) \frac{\partial u_{t_k,x}}{\partial y}(\theta,z) \right) dz d\theta.$$

$|B|$ and $|\frac{\partial B}{\partial x}|$ being uniformly bounded, we have

$$\begin{aligned}\left|\frac{\partial^2 u_{t_k,x}}{\partial y^2}\right|(s,y) &\le |g'_{\sigma^2 s+\varepsilon}|(y-x) + C\int_0^s\int_{\mathbb{R}} |g'_{\sigma^2(s-\theta)}|(y-z)g_{\varepsilon+2\sigma^2\theta}(x-z)dzd\theta \\ &\quad + C\int_0^s\int_{\mathbb{R}} |g'_{\sigma^2(s-\theta)}|(y-z)|\frac{\partial^2 u_{t_k,x}}{\partial y^2}|(\theta,z)dzd\theta \\ &\le |g'_{\sigma^2 s+\varepsilon}|(y-x) + Cg_{2\sigma^2 s+\varepsilon}(y-x) \\ &\quad + C\int_0^s\int_{\mathbb{R}} |g'_{\sigma^2(s-\theta)}|(y-z)|\frac{\partial^2 u_{t_k,x}}{\partial y^2}|(\theta,z)dzd\theta \end{aligned} \tag{3.15}$$

and hence

$$\sup_{y\in\mathbb{R}}\int_{\mathbb{R}}\left|\frac{\partial^2 u_{t_k,x}}{\partial y^2}(s,y)\right|dx \le \frac{C}{\sqrt{\varepsilon+\sigma^2 s}} + \int_0^s \frac{C}{\sqrt{\varepsilon+\sigma^2(s-\theta)}}\sup_{z\in\mathbb{R}}\int_{\mathbb{R}}\left|\frac{\partial^2 u_{t_k,x}}{\partial y^2}(\theta,z)\right|dxd\theta.$$

We apply Gronwall's lemma to get (3.13). To prove (3.14), we start from (3.15):

$$\begin{aligned}\left|\frac{\partial^2 u_{t_k,x}}{\partial y^2}\right|^{5/4}(s,y) &\le C|g'_{\sigma^2 s+\varepsilon}|^{5/4}(y-x) + Cg^{5/4}_{2\sigma^2 s+\varepsilon}(y-x) \\ &\quad + \left(C\int_0^s\int_{\mathbb{R}} |g'_{\sigma^2(s-\theta)}|(y-z)|\frac{\partial^2 u_{t_k,x}}{\partial y^2}|(\theta,z)dzd\theta\right)^{5/4} \\ &\le \frac{C}{(\sigma^2 s+\varepsilon)^{3/4}}g_{2\sigma^2 s+2\varepsilon}(y-x) \\ &\quad + \int_0^s\int_{\mathbb{R}}\frac{C}{(s-\theta)^{5/8}}\left|\frac{\partial^2 u_{t_k,x}}{\partial y^2}\right|^{5/4}(\theta,z)g_{2\sigma^2(s-\theta)}(y-z)dzd\theta.\end{aligned}$$

Hence,

$$\int_{\mathbb{R}}\left|\frac{\partial^2 u_{t_k,x}}{\partial y^2}\right|^{5/4}(s,y)dy \le \frac{C}{(\sigma^2 s+\varepsilon)^{3/4}} + \int_0^s\frac{C}{(s-\theta)^{5/8}}\int_{\mathbb{R}}\left|\frac{\partial^2 u_{t_k,x}}{\partial y^2}\right|^{5/4}(\theta,z)dzd\theta,$$

from which we conclude by Gronwall's lemma. ∎

3.2 Estimates on the smoothing error

Proof of Lemma 3.1: First we observe that for all $x\in\mathbb{R}$, $H_\varepsilon(x)=\mathbb{E}H(x-W_\varepsilon)$. Then for any $z\in\mathbb{R}$,

$$\int_{\mathbb{R}}|H(x-z)-H_\varepsilon(x-z)|dx \le \mathbb{E}\int_{\mathbb{R}}|H(x-z)-H(x-z-W_\varepsilon)|dx = \mathbb{E}|W_\varepsilon| = \frac{2\sqrt{\varepsilon}}{\sqrt{2\pi}}.$$

The density $U(t,z)$ of Z_t being uniformly bounded in $z \in \mathbb{R}$,

$$|\mathbb{E}H(x - Z_{t_k}) - \mathbb{E}H_\varepsilon(x - Z_{t_k})| \leq \int_{\mathbb{R}} |H(x-z) - H_\varepsilon(x-z)|U(t_k, z)dz \leq C\sqrt{\varepsilon},$$

which gives (3.7). Now for any $x \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}|H(x - Y^i_{t_{k+1}}) - H_\varepsilon(x - Y^i_{t_{k+1}})| \\
&= \mathbb{E}\left(\mathbb{E}^{\mathcal{F}_{t_k}} |H(x - Y^i_{t_k} - \Delta t A'(\overline{V}_{t_k}(Y^i_{t_k})) - \sigma W_{\Delta t}) - H_\varepsilon(x - Y^i_{t_k} - \Delta t A'(\overline{V}_{t_k}(Y^i_{t_k})) - \sigma W_{\Delta t})|\right) \\
&= \int_{\mathbb{R}} g_{\sigma^2\Delta t}(z)\mathbb{E}|H(x - Y^i_{t_k} - \Delta t A'(\overline{V}_{t_k}(Y^i_{t_k})) - z) - H_\varepsilon(x - Y^i_{t_k} - \Delta t A'(\overline{V}_{t_k}(Y^i_{t_k})) - z)|dz \\
&\leq C\frac{\sqrt{\varepsilon}}{\sqrt{\Delta t}},
\end{aligned}$$

from which we deduce (3.8). ■

3.3 Estimates on the initialization error

Proof of Lemma 3.2: For all $t > 0$, the function $y \longrightarrow \mathbb{E}H_\varepsilon(x - Z^{0,y}_t)$ is differentiable and

$$\frac{\partial}{\partial y}\mathbb{E}H_\varepsilon(x - Z^{0,y}_t) = \mathbb{E}\left(g_\varepsilon(x - Z^{0,y}_t)\frac{dZ^{0,y}_t}{dy}\right),$$

where $\frac{d}{dy}Z^{0,y}_t = \exp\left(\int_0^t \frac{\partial B}{\partial x}(s, Z^{0,y}_s)ds\right) \leq C$. Thus by integration by parts,

$$\begin{aligned}
\int_{\mathbb{R}} \mathbb{E}H_\varepsilon(x - Z^{0,y}_t)m_0(dy) \quad &= \quad \mathbb{E}H_\varepsilon(x - Z^{0,0}_t) - \int_{-\infty}^0 \frac{\partial}{\partial y}\mathbb{E}H_\varepsilon(x - Z^{0,y}_t)V_0(y)dy \\
&\quad + \int_0^{+\infty} \frac{\partial}{\partial y}\mathbb{E}H_\varepsilon(x - Z^{0,y}_t)(1 - V_0(y))dy.
\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}} \mathbb{E}H_\varepsilon(x - Z^{0,y}_t)\overline{m}_0(dy) = \int_{-\infty}^0 \mathbb{E}H_\varepsilon(x - Z^{0,y}_t)\ d\overline{V}_0(y) \ - \ \int_0^{+\infty} \mathbb{E}H_\varepsilon(x - Z^{0,y}_t)\ d(1 - d\overline{V}_0(y))$$

and the integration by parts formula for a Stieljes integral gives

$$\begin{aligned}
\int_{\mathbb{R}} \mathbb{E}H_\varepsilon(x - Z^{0,y}_t)\overline{m}_0(dy) \quad &= \quad \mathbb{E}H_\varepsilon(x - Z^{0,0}_t) - \int_{-\infty}^0 \frac{\partial}{\partial y}\mathbb{E}H_\varepsilon(x - Z^{0,y}_t)\overline{V}_0(y)dy \\
&\quad + \int_0^{+\infty} \frac{\partial}{\partial y}\mathbb{E}H_\varepsilon(x - Z^{0,y}_t)(1 - \overline{V}_0(y))dy.
\end{aligned}$$

Hence, we obtain the following expression for the initialization error

$$\int_{\mathbb{R}} \mathbb{E}H_\varepsilon(x - Z_t^{0,y})m_0(dy) - \int_{\mathbb{R}} \mathbb{E}H_\varepsilon(x - Z_t^{0,y})\overline{m}_0(dy) = \int_{\mathbb{R}} \frac{\partial}{\partial y}\mathbb{E}H_\varepsilon(x - Z_t^{0,y})(\overline{V}_0(y) - V_0(y))dy,$$

from which we easily deduce that

$$\sup_{x\in\mathbb{R}}\left|\int_{\mathbb{R}} \mathbb{E}H_\varepsilon(x - Z_{t_k}^{y,0})m_0(dy) - \int_{\mathbb{R}} \mathbb{E}H_\varepsilon(x - Z_{t_k}^{y,0})\overline{m}_0(dy)\right| \\ \leq C\|V_0 - \overline{V}_0\|_{L^\infty(\mathbb{R})}\sup_{x\in\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}g_\varepsilon(x - Z_{t_k}^{0,y})dy$$

and

$$\left\|\int_{\mathbb{R}} \mathbb{E}H_\varepsilon(x - Z_{t_k}^{y,0})m_0(dy) - \int_{\mathbb{R}} \mathbb{E}H_\varepsilon(x - Z_{t_k}^{y,0})\overline{m}_0(dy)\right\|_{L^1(\mathbb{R})} \\ \leq C\|V_0 - \overline{V}_0\|_{L^1(\mathbb{R})}\sup_{y\in\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}g_\varepsilon(x - Z_{t_k}^{0,y})dx.$$

In view of Lemma 3.8, the exponential bound for the density of $Z_t^{0,y}$ gives

$$\mathbb{E}g_\varepsilon(x - Z_{t_k}^{0,y}) \leq C g_{\varepsilon+2\sigma^2 t_k}(x - y), \tag{3.16}$$

from which we easily conclude. ∎

3.4 Estimates on the statistical error

Proof of Lemma 3.3: We consider the statistical error

$$\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^N \int_0^{t_k} \sigma\frac{\partial v_{t_k,x}}{\partial y}(s, Y_s^i)dW_s^i\right|.$$

From (3.12), $\frac{\partial v_{t_k,x}}{\partial y}(s,y)$ is uniformly bounded on $[0,t_k]\times\mathbb{R}$ by $C/\sqrt{\varepsilon}$. Then, by Cauchy-Schwarz's inequality, for all $x\in\mathbb{R}$,

$$\begin{aligned}\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^N \int_0^{t_k} \sigma\frac{\partial v_{x,t_k}}{\partial y}(s, Y_s^i)dW_s^i\right| &\leq \sqrt{\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^N \int_0^{t_k} \sigma\frac{\partial v_{x,t_k}}{\partial y}(s, Y_s^i)dW_s^i\right)^2} \\ &\leq \sqrt{\frac{1}{N^2}\sum_{i=1}^N \int_0^{t_k} \sigma^2\mathbb{E}\left(\frac{\partial v_{x,t_k}}{\partial y}(s, Y_s^i)\right)^2 ds.}\end{aligned}$$

(Y^i) being the solution of Equation (3.1), for each i in $\{1,\dots,N\}$, let $(\mathcal{Z}^i)$ be the exponential martingale

$$\mathcal{Z}_t^i = \exp\left(\int_0^t \frac{A'(\overline{V}_{\eta(s)}(Y_{\eta s}^i))}{\sigma} dW_s^i - \frac{1}{2}\int_0^t \frac{(A'(\overline{V}_{\eta(s)}(Y_{\eta s}^i)))^2}{\sigma^2} ds\right). \tag{3.17}$$

where $\eta(t) = \sup_{k\in\{0,\dots,K\}} \{t_k; t_k \leq t\}$. Then, by Girsanov's theorem and Estimate (3.12),

$$\begin{aligned}
\mathbb{E}\left(\frac{\partial v_{x,t_k}}{\partial y}(s, Y_s^i)\right)^2 &= \mathbb{E}\left[\left(\frac{\partial v_{x,t_k}}{\partial y}(s, y_0^i + \sigma W_s^i)\right)^2 \mathcal{Z}_s^i\right] \\
&\leq \sqrt{\mathbb{E}\left(\frac{\partial v_{x,t_k}}{\partial y}(s, y_0^i + \sigma W_s^i)\right)^4} \exp\left(\frac{s}{2\sigma^2}\sup_{v\in[0,1]}|A'(v)|^2\right) \\
&\leq C\sqrt{\left(g^4_{\varepsilon+2\sigma^2(t_k-s)} * g_{\sigma^2 s}\right)(x - y_0^i)}.
\end{aligned}$$

An easy computation shows that, for any $z \in \mathbb{R}$,

$$g^4_{\varepsilon+2\sigma^2(t_k-s)} * g_{\sigma^2 s}(z) \leq \frac{C}{\sqrt{t_k}(t_k - s)^{3/2}} \exp\left(-\frac{4z^2}{2(\varepsilon + 4\sigma^2 t_k)}\right).$$

Finally for all $x \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^N \int_0^{t_k} \sigma \frac{\partial v_{x,t_k}}{\partial y}(s, Y_s^i) dW_s^i\right| &\leq \frac{C}{\sqrt{N}}\sqrt{\frac{1}{N}\sum_{i=1}^N \exp\left(-\frac{(x-y_0^i)^2}{(\varepsilon+4\sigma^2 t_k)}\right)\frac{1}{{t_k}^{1/4}}\int_0^{t_k}\frac{1}{(t_k-s)^{3/4}}} \\
&\leq \frac{C}{\sqrt{N}}\sqrt{\frac{1}{N}\sum_{i=1}^N \exp\left(-\frac{(x-y_0^i)^2}{(\varepsilon+4\sigma^2 t_k)}\right)}.
\end{aligned}$$

Thus,

$$\sup_{x\in\mathbb{R}} \mathbb{E}\left|\frac{1}{N}\sum_{i=1}^N \int_0^{t_k} \sigma \frac{\partial v_{t_k,x}}{\partial y}(s, Y_s^i) dW_s^i\right| \leq C/\sqrt{N}$$

and

$$\mathbb{E}\left\|\frac{1}{N}\sum_{i=1}^N \int_0^{t_k} \sigma \frac{\partial v_{t_k,\cdot}}{\partial y}(s, Y_s^i) dW_s^i\right\|_{L^1(\mathbb{R})} \leq \frac{C}{\sqrt{N}}\int_{\mathbb{R}}\sqrt{\frac{1}{N}\sum_{i=1}^N \exp\left(-\frac{(x-y_0^i)^2}{(\varepsilon+4\sigma^2 t_k)}\right)}\, dx.$$

We set $\phi(x) = \exp\left(-\frac{x^2}{(\varepsilon+4\sigma^2 t_k)}\right)$. The above integral becomes $\int_{\mathbb{R}}\sqrt{\phi * \overline{m}_0(x)}dx$. Under hypothesis (H2′), by construction all the initial positions $(y_0^i, i = 1,\dots,N)$ are in the interval

$[y_0^1, y_0^N]$. To end the proof, we decompose the integral into three parts and remark that

$$\begin{aligned}\int_{\mathbb{R}} \sqrt{\phi * \overline{m}_0(x)}dx &= \int_{-\infty}^{y_0^1} \sqrt{\phi * \overline{m}_0(x)}dx + \int_{y_0^1}^{y_0^N} \sqrt{\phi * \overline{m}_0(x)}dx + \int_{y_0^N}^{+\infty} \sqrt{\phi * \overline{m}_0(x)}dx \\ &\leq \int_{y_0^1}^{y_0^N} \sqrt{\phi * \overline{m}_0(x)}dx + 2\int_{-\infty}^{0} \exp\left(-\frac{x^2}{2(\varepsilon + 4\sigma^2 t_k)}\right)dx \\ &\leq \int_{y_0^1}^{y_0^N} \sqrt{\phi * \overline{m}_0(x)}dx + C.\end{aligned}$$

Now we use the hypothesis (H3), in particular we introduce the positive constant M such that $U_0(x) \leq \eta \exp(-\alpha \frac{x^2}{2})$ for $|x| > M$. If y_0^1 and y_0^N are both in the intervale $[-M, M]$, then

$$\int_{y_0^1}^{y_0^N} \sqrt{\phi * \overline{m}_0(x)}dx \leq 2M.$$

In the other case, we use the integration by parts formula to get

$$\phi * \overline{m}_0(x) = \phi * U_0(x) + \left(\phi' * (V_0 - \overline{V}_0)\right)(x).$$

We treat the case $y_0^1 < -M < M < y_0^N$ only, the other cases can be treated with similar arguments. We have

$$\int_{y_0^1}^{y_0^N} \sqrt{\phi * \overline{m}_0(x)}dx \leq \int_{y_0^1}^{y_0^N} \sqrt{\phi * U_0(x)}dx + \int_{y_0^1}^{y_0^N} \sqrt{\left(\phi' * (V_0 - \overline{V}_0)\right)(x)}dx.$$

For the first term in the right-hand side,

$$\int_{y_0^1}^{y_0^N} \sqrt{\phi * U_0(x)}dx \leq \int_{-\infty}^{-M} \sqrt{\phi * \left(\eta \exp(-\alpha \frac{(\cdot)^2}{2}\right)(x)}dx + 2M + \int_{M}^{+\infty} \sqrt{\phi * \left(\eta \exp(-\alpha \frac{(\cdot)^2}{2}\right)(x)}dx$$

and

$$\phi * \left(\eta \exp(-\alpha \frac{(\cdot)^2}{2}\right)(x) \leq \eta\sqrt{\pi\alpha} \exp\left(-\frac{\alpha x^2}{2(1 + \alpha(\varepsilon + 4\sigma^2 t_k)}\right),$$

which implies that $\int_{-\infty}^{-M} \sqrt{\phi * \left(\eta \exp(-\alpha \frac{(\cdot)^2}{2}\right)(x)}dx \leq C$ and thus $\int_{y_0^1}^{y_0^N} \sqrt{\phi * U_0(x)}dx \leq C$ (the constant C depends on T but does not depend on t_k). For the second term in the right-hand side, as $\|V_0 - \overline{V}_0\|_{L^\infty(\mathbb{R})} \leq 1/N$ and $\|\phi'\|_{L^1(\mathbb{R})} \leq C$ (independent of t_k), then

$$\int_{y_0^1}^{y_0^N} \sqrt{\left(\phi' * (V_0 - \overline{V}_0)\right)(x)}dx \leq C(|y_0^1| + |y_0^N|)\sqrt{1/N}.$$

Thanks to hypothesis (H3), it is not difficult to see that $(|y_0^1| + |y_0^N|) \leq C\sqrt{\ln(N)}$, which concludes the proof. ∎

3.5 Proof of Lemma 3.4: estimates for the discretization error

We consider now the main part of the error in the decomposition (3.6). We split it into two parts, in order to make appear the difference of the drift functions B at the discrete times t_l and its approximation $A'(\overline{V}_{t_l}(x))$:

$$\begin{aligned}
&\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N}\int_0^{t_k}\frac{\partial v_{t_k,x}}{\partial y}(s,Y_s^i)\left[B(s,Y_s^i)-A'\left(\overline{V}_{\eta(s)}(Y_{\eta(s)}^i)\right)\right]ds\right|\\
&\leq \mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N}\sum_{l=0}^{k-1}\int_{t_l}^{t_{l+1}}\frac{\partial v_{t_k,x}}{\partial y}(s,Y_s^i)\left[B(s,Y_s^i)-B(t_l,Y_{t_l}^i)\right]ds\right|\\
&+\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N}\sum_{l=0}^{k-1}\int_{t_l}^{t_{l+1}}\frac{\partial v_{t_k,x}}{\partial y}(s,Y_s^i)\left[B(t_l,Y_{t_l}^i)-A'\left(\overline{V}_{t_l}(Y_{t_l}^i)\right)\right]ds\right|\\
&:= T_1(x)+T_2(x).
\end{aligned}$$

We treat $T_1(x)$ and $T_2(x)$ separately.

Upper bound for $T_1(x)$: this first term is a time discretization error. In order to obtain an error bound of order $\mathcal{O}(\Delta t)$, we need to make appear an expectation inside the absolute value in the expression of $T_1(x)$. For all $l \in \{0,\ldots,K\}$, we set $\mathcal{F}_{t_l} = \sigma(W_s^i; 0 \leq s \leq t_l, i = 1,\ldots,N)$. For all $s \in [t_l, t_{l+1})$, the variables $(R_{t_l,s}^i := \frac{\partial v_{t_k,x}}{\partial y}(s,Y_s^i)\left[B(s,Y_s^i) - B(t_l,Y_{t_l}^i)\right], i = 1,\ldots,N)$ are $\mathcal{F}_{t_l}$-conditionally independent. Hence,

$$\begin{aligned}
\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N}R_{t_l,s}^i-\mathbb{E}^{\mathcal{F}_{t_l}}\left(R_{t_l,s}^i\right)\right| &\leq \frac{1}{\sqrt{N}}\sqrt{\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left(R_{t_l,s}^i\right)^2}\\
&\leq \frac{C}{\sqrt{N}}\sqrt{\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left(\frac{\partial v_{t_k,x}}{\partial y}(s,Y_s^i)\right)^2}.
\end{aligned}$$

Thus, we isolate a statistical error in $T_1(x)$:

$$\begin{aligned}
T_1(x) \quad\leq\quad & \mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N}\sum_{l=0}^{k-1}\int_{t_l}^{t_{l+1}}\mathbb{E}^{\mathcal{F}_{t_l}}\left\{\frac{\partial v_{t_k,x}}{\partial y}(s,Y_s^i)\left[B(s,Y_s^i)-B(t_l,Y_{t_l}^i)\right]\right\}ds\right|\\
&+\frac{C}{\sqrt{N}}\int_0^{t_k}\sqrt{\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left(\frac{\partial v_{t_k,x}}{\partial y}(s,Y_s^i)\right)^2}ds.
\end{aligned}$$

By Itô's formula,

$$
\begin{aligned}
&\mathbb{E}^{\mathcal{F}_{t_l}}\left\{\frac{\partial v_{t_k,x}}{\partial y}(s,Y_s^i)\left[B(s,Y_s^i)-B(t_l,Y_{t_l}^i)\right]\right\}\\
&=\mathbb{E}^{\mathcal{F}_{t_l}}\int_{t_l}^{s}\left(\frac{\partial}{\partial\theta}+A'\left(\overline{V}_{t_l}(Y_{t_l}^i)\right)\frac{\partial}{\partial y}+\frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial v_{t_k,x}}{\partial y}(B-B(t_l,Y_{t_l}^i))\right)(\theta,Y_\theta^i)d\theta\\
&=\mathbb{E}^{\mathcal{F}_{t_l}}\int_{t_l}^{s}\left[\frac{\partial v_{t_k,x}}{\partial y}\left(\frac{\partial B}{\partial\theta}+\frac{\sigma^2}{2}\frac{\partial^2 B}{\partial y^2}+A'\left(\overline{V}_{t_l}(Y_{t_l}^i)\right)\frac{\partial B}{\partial y}\right)\right.\\
&\qquad\left.+\left(\sigma^2\frac{\partial B}{\partial y}+(B-B(t_l,Y_{t_l}^i))(A'\left(\overline{V}_{t_l}(Y_{t_l}^i)\right)-B\right)\frac{\partial^2 v_{t_k,x}}{\partial y^2}\right](\theta,Y_\theta^i)d\theta.
\end{aligned}
$$

The last identity is obtained by using Equation (3.9). As $B(s,y)$ has uniformly bounded derivatives, we get

$$
\begin{aligned}
&\mathbb{E}\int_{t_l}^{t_{l+1}}\left|\mathbb{E}^{\mathcal{F}_{t_l}}\left\{\frac{\partial v_{t_k,x}}{\partial y}(s,Y_s^i)\left[B(s,Y_s^i)-B(t_l,Y_{t_l}^i)\right]\right\}\right|ds\\
&\leq C\int_{t_l}^{t_{l+1}}\int_{t_l}^{s}\mathbb{E}\left[\left|\frac{\partial v_{t_k,x}}{\partial y}\right|(\theta,Y_\theta^i)+\left|\frac{\partial^2 v_{t_k,x}}{\partial y^2}\right|(\theta,Y_\theta^i)\right]d\theta ds\\
&\leq C\Delta t\int_{t_l}^{t_{l+1}}\mathbb{E}\left[\left|\frac{\partial v_{t_k,x}}{\partial y}\right|(s,Y_s^i)+\left|\frac{\partial^2 v_{t_k,x}}{\partial y^2}\right|(s,Y_s^i)\right]ds,
\end{aligned}
$$

and

$$
\begin{aligned}
T_1(x)\leq &C\Delta t\int_0^{t_k}\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\left(\left|\frac{\partial v_{t_k,x}}{\partial y}\right|+\left|\frac{\partial^2 v_{t_k,x}}{\partial y^2}\right|\right)(s,Y_s^i)\right]ds\\
&+\frac{C}{\sqrt{N}}\int_0^{t_k}\sqrt{\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left(\frac{\partial v_{t_k,x}}{\partial y}(s,Y_s^i)\right)^2}ds.
\end{aligned}
$$

Now we need to estimate $\|T_1(\cdot)\|_{L^1(\mathbb{R})}$ and $\sup_{x\in\mathbb{R}}T_1(x)$. We easily deduce from the proof of Lemma 3.3 that

$$
\sup_{x\in\mathbb{R}}\left[\int_0^{t_k}\sqrt{\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left(\frac{\partial v_{t_k,x}}{\partial y}(s,Y_s^i)\right)^2}\right]+\int_{\mathbb{R}}\int_0^{t_k}\sqrt{\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left(\frac{\partial v_{t_k,x}}{\partial y}(s,Y_s^i)\right)^2}dx
$$

is bounded by a constant C depending only in σ, T, A and V_0. Moreover, by Lemma 3.9,

$$
\int_{\mathbb{R}}\left[\left|\frac{\partial v_{t_k,x}}{\partial y}\right|(s,Y_s^i)+\left|\frac{\partial^2 v_{t_k,x}}{\partial y^2}\right|(s,Y_s^i)\right]dx\leq\frac{C}{\sqrt{t_k-s}}.
$$

Hence, we obtain that

$$
\|T_1(\cdot)\|_{L^1(\mathbb{R})}\leq C\left(\Delta t+\frac{1}{\sqrt{N}}\right). \tag{3.18}
$$

Still by Lemma 3.9, we observe that $\sup_{x\in\mathbb{R}}\left|\frac{\partial v_{t_k,x}}{\partial y}\right|(s,Y^i_s)\leq \frac{C}{\sqrt{t_k-s}}$. It remains to bound $\sup_{x\in\mathbb{R}}\mathbb{E}\left[\left|\frac{\partial^2 v_{t_k,x}}{\partial y^2}\right|(s,Y^i_s)\right]$. Let $(\mathcal{Z}^i)$ be the exponential martingale defined in (3.17). By Girsanov's theorem,

$$\begin{aligned}
\mathbb{E}\left[\left|\frac{\partial^2 v_{t_k,x}}{\partial y^2}\right|(s,Y^i_s)\right] &= \mathbb{E}\left[\mathcal{Z}^i_s\left|\frac{\partial^2 v_{t_k,x}}{\partial y^2}\right|(s,y^i_0+\sigma W^i_s)\right]\\
&\leq C\left(\mathbb{E}\left|\frac{\partial^2 v_{t_k,x}}{\partial y^2}\right|^{5/4}(s,y^i_0+\sigma W^i_s)\right)^{4/5}\\
&\leq \frac{C}{s^{4/10}}\left\|\left|\frac{\partial^2 v_{t_k,x}}{\partial y^2}\right|^{5/4}(s,\cdot)\right\|_{L^1(\mathbb{R})}.
\end{aligned}\tag{3.19}$$

Using (3.14), we obtain that $\mathbb{E}\left[\left|\frac{\partial^2 v_{t_k,x}}{\partial y^2}\right|(s,Y^i_s)\right]\leq\frac{C}{s^{4/10}(t_k-s)^{3/5}}$ and hence,

$$\sup_{x\in\mathbb{R}}T_1(x)\leq C\left(\Delta t+\frac{1}{\sqrt{N}}\right).\tag{3.20}$$

Upper bound for $T_2(x)$: for all (t,x), $B(t,x)=A'(V(t,x))=A'(\mathbb{E}H(x-Z_t))$. Hence,

$$T_2(x)\leq\sup_{v\in[0,1]}|A'(v)|\frac{1}{N}\sum_{i=1}^N\sum_{l=0}^{k-1}\mathbb{E}\int_{t_l}^{t_{l+1}}\left|\frac{\partial v_{t_k,x}}{\partial y}(s,Y^i_s)\right|\left|V(t_l,Y^i_{t_l})-\overline{V}_{t_l}(Y^i_{t_l})\right|ds.$$

By Lemma 3.9, $\sup_{x\in\mathbb{R}}\|\dfrac{\partial v_{t_k,x}}{\partial z}(s,\cdot)\|_{L^\infty(\mathbb{R})}+\sup_{z\in\mathbb{R}}\|\dfrac{\partial v_{t_k,\cdot}}{\partial z}(s,z)\|_{L^1(\mathbb{R})}$ is bounded by $\leq C/\sqrt{t_k-s}$. Then,

$$\sup_{x\in\mathbb{R}}\mathbb{E}T_2(x)+\mathbb{E}\|T_2(\cdot)\|_{L^1(\mathbb{R})}\leq\sum_{l=0}^{k-1}\left(\int_{t_l}^{t_{l+1}}\frac{C}{\sqrt{t_k-s}}ds\right)\frac{1}{N}\sum_{i=1}^N\mathbb{E}\left|V(t_l,Y^i_{t_l})-\overline{V}_{t_l}(Y^i_{t_l})\right|.\tag{3.21}$$

Now the estimation of T_2 is based on the upper bound of terms of the sequence

$$\left(\frac{1}{N}\sum_{i=1}^N\mathbb{E}\left|V(t_l,Y^i_{t_l})-\overline{V}_{t_l}(Y^i_{t_l})\right|\right)_{l=1,\dots,k-1}.$$

To obtain an induction formula on this sequence we introduce a new family of discrete time processes. For each $i\in\{1,\dots,N\}$, we denote by $(\overline{Z}^i_{t_k},k=0,\dots,K)$ the discrete-time process solution of

$$\begin{cases}\overline{Z}^i_{t_{k+1}}=\overline{Z}^i_{t_k}+\Delta tB(t_k,\overline{Z}^i_{t_k})+\sigma(W^i_{t_{k+1}}-W^i_{t_k}),\\ \overline{Z}^i_0=y^i_0,\end{cases}$$

and for all $y \in \mathbb{R}$, we denote by $(\overline{Z}^{0,y}_{t_k}, k = 0, \ldots, K)$ the discrete-time process solution of

$$\left\{ \begin{array}{l} \overline{Z}^{0,y}_{t_{k+1}} = \overline{Z}^{0,y}_{t_k} + \Delta t B(t_k, \overline{Z}^{0,y}_{t_k}) + \sigma(W_{t_{k+1}} - W_{t_k}), \\ \overline{Z}^{0,y}_0 = y. \end{array} \right.$$

Similarly, for all $y \in \mathbb{R}$ and all $l \in \{0, \ldots, K\}$ we denote by $(\overline{Z}^{i,t_l,y}_{t_k}, k = l, \ldots, K)$ the discrete-time process solution of

$$\left\{ \begin{array}{l} \overline{Z}^{i,t_l,y}_{t_{k+1}} = \overline{Z}^{i,t_l,y}_{t_k} + \Delta t B(t_k, \overline{Z}^{i,t_l,y}_{t_k}) + \sigma(W^i_{t_{k+1}} - W^i_{t_k}), \\ \overline{Z}^{i,t_l,y}_{t_l} = y. \end{array} \right.$$

The function V being uniformly Lipschitz, we remark that

$$\begin{aligned} \left| V(t_l, Y^i_{t_l}) - V(t_l, \overline{Z}^i_{t_l}) \right| \leq C \left| Y^i_{t_l} - \overline{Z}^i_{t_l} \right| & \leq C \left| \Delta t \sum_{m=0}^{l-1} A'(\overline{V}_{t_m}(Y^i_{t_m}) - A'(V(t_m, \overline{Z}^i_{t_m})) \right| \\ & \leq C \Delta t \sum_{m=0}^{l-1} \left| V(t_m, \overline{Z}^i_{t_m}) - \overline{V}_{t_m}(Y^i_{t_m}) \right|. \end{aligned}$$

Then,

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left| V(t_l, Y^i_{t_l}) - \overline{V}_{t_l}(Y^i_{t_l}) \right| \leq & \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left| V(t_l, \overline{Z}^i_{t_l}) - \overline{V}_{t_l}(Y^i_{t_l}) \right| \\ & + C \Delta t \sum_{m=0}^{l-1} \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left| V(t_m, \overline{Z}^i_{t_m}) - \overline{V}_{t_m}(Y^i_{t_m}) \right|. \end{aligned}$$

For all $l \in \{1, \ldots, K\}$, we set

$$\overline{E}_{t_l} := \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left| V(t_l, \overline{Z}^i_{t_l}) - \overline{V}_{t_l}(Y^i_{t_l}) \right|.$$

Thus we have

$$\frac{1}{N} \sum_{i=1}^N \mathbb{E} \left| V(t_l, Y^i_{t_l}) - \overline{V}_{t_l}(Y^i_{t_l}) \right| \leq \overline{E}_{t_l} + C \Delta t \sum_{m=0}^{l-1} \overline{E}_{t_m}. \tag{3.22}$$

An induction relation for $(\overline{E}_{t_l}, l = 0, \ldots, K)$ is given in the following

Lemma 3.10 *For $l = 0, \ldots, K$, one has*

$$\overline{E}_{t_l} \leq \sum_{n=0}^{l-1} \frac{C \Delta t}{\sqrt{t_l - t_n}} \overline{E}_{t_n} + C \left(\Delta t + \|V_0 - \overline{V}_0\|_{L^\infty(\mathbb{R})} + \frac{1}{\sqrt{N}} \right)$$

and by Gronwall's lemma,

$$\overline{E}_{t_l} \leq C\left(\Delta t + \|V_0 - \overline{V}_0\|_{L^\infty(\mathbb{R})} + \frac{1}{\sqrt{N}}\right).$$

In view of (3.21), (3.22) and this previous estimate, we obtain that

$$\begin{aligned}\sup_{x\in\mathbb{R}} \mathbb{E}T_2(x) + \mathbb{E}\|T_2(\cdot)\|_{L^1(\mathbb{R})} &\leq \sum_{l=0}^{k-1} \frac{C\Delta t}{\sqrt{t_k - t_l}}\left(\Delta t + \|V_0 - \overline{V}_0\|_{L^\infty(\mathbb{R})} + \frac{1}{\sqrt{N}}\right)\\ &\leq C\left(\|V_0 - \overline{V}_0\|_{L^\infty(\mathbb{R})} + \Delta t + \frac{1}{\sqrt{N}}\right).\end{aligned}$$

In view of Estimates (3.18) and (3.20) on T_1, this ends the proof of Lemma 3.4.

Proof of Lemma 3.10: First, we note that $\overline{E}_0 \leq \|V_0 - \overline{V}_0\|_{L^\infty(\mathbb{R})}$ and for $l = 1, \ldots, K$,

$$\overline{E}_{t_l} = \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left|\mathbb{E}H(x - Z_{t_l})\Big|_{x=\overline{Z}^i_{t_l}} - \frac{1}{N}\sum_{j=1}^N H(Y^i_{t_l} - Y^j_{t_l})\right|.$$

To prove the induction formula, we decompose each term $\overline{E}_{t_l}$ in five parts. As in the beginning of the proof of Theorem 2.2, we make appear a smoothing error, an initialization error, a discretization error and a statistical error. First, we introduce the artificial smoothing of the Heaviside function:

$$\begin{aligned}\overline{E}_{t_l} \quad\leq\quad & \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left|\mathbb{E}H_\varepsilon(x - Z_{t_l})\Big|_{x=\overline{Z}^i_{t_l}} - \frac{1}{N}\sum_{j=1}^N H_\varepsilon(Y^i_{t_l} - Y^j_{t_l})\right|\\ &+\frac{1}{N}\sum_{i=1}^N \left|\mathbb{E}H_\varepsilon(x - Z_{t_l})\Big|_{x=\overline{Z}^i_{t_l}} - \mathbb{E}H(x - Z_{t_l})\Big|_{x=\overline{Z}^i_{t_l}}\right|\\ &+\frac{1}{N}\sum_{i=1}^N \mathbb{E}\left|\frac{1}{N}\sum_{j=1}^N H_\varepsilon(Y^i_{t_l} - Y^j_{t_l}) - \frac{1}{N}\sum_{j=1}^N H(Y^i_{t_l} - Y^j_{t_l})\right|\end{aligned}$$

and by Lemma 3.1,

$$\overline{E}_{t_l} \leq \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left|\mathbb{E}H_\varepsilon(x - Z_{t_l})\Big|_{x=\overline{Z}^i_{t_l}} - \frac{1}{N}\sum_{j=1}^N H_\varepsilon(Y^i_{t_l} - Y^j_{t_l})\right| + C\frac{\sqrt{\varepsilon}}{\sqrt{\Delta t}}.$$

We choose $\varepsilon \leq \Delta t^3$. The next step consists in introducing the initialization error:

$$\begin{aligned}
\overline{E}_{t_l} \quad \leq \quad & \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|\frac{1}{N}\sum_{j=1}^{N}\left(\mathbb{E}H_\varepsilon(x - Z_{t_l}^{0,y_0^j})\Big|_{x=\overline{Z}_{t_l}^i} - H_\varepsilon(Y_{t_l}^i - Y_{t_l}^j)\right)\right| \\
& + \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|\frac{1}{N}\sum_{j=1}^{N}\mathbb{E}H_\varepsilon(x - Z_{t_l}^{0,y_0^j})\Big|_{x=\overline{Z}_{t_l}^i} - \mathbb{E}H_\varepsilon(x - Z_{t_l})\Big|_{x=\overline{Z}_{t_l}^i}\right| + C\Delta t.
\end{aligned}$$

Following the same technique than in the proof of Lemma 3.2, we get

$$\begin{aligned}
& \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|\frac{1}{N}\sum_{j=1}^{N}\mathbb{E}H_\varepsilon(x - Z_{t_l}^{0,y_0^j})\Big|_{x=\overline{Z}_{t_l}^i} - \mathbb{E}H_\varepsilon(x - Z_{t_l})\Big|_{x=\overline{Z}_{t_l}^i}\right| \\
& \leq \sup_{x\in\mathbb{R}}\left|\int_{\mathbb{R}}\mathbb{E}H_\varepsilon(x - Z_t^{0,y})\overline{m}_0(dy) - \int_{\mathbb{R}}\mathbb{E}H_\varepsilon(x - Z_t^{0,y})m_0(dy)\right| \\
& \leq C\|V_0 - \overline{V}_0\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

Our third step consists in making appear the Euler scheme error:

$$\begin{aligned}
\overline{E}_{t_l} \quad \leq \quad & \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|\frac{1}{N}\sum_{j=1}^{N}\left(\mathbb{E}H_\varepsilon(x - \overline{Z}_{t_l}^{0,y_0^j})\Big|_{x=\overline{Z}_{t_l}^i} - H_\varepsilon(Y_{t_l}^i - Y_{t_l}^j)\right)\right| \\
& + \frac{1}{N^2}\sum_{i,j=1}^{N}\left|\mathbb{E}H_\varepsilon(x - Z_{t_l}^{0,y_0^j})\Big|_{x=\overline{Z}_{t_l}^i} - \mathbb{E}H_\varepsilon(x - \overline{Z}_{t_l}^{0,y_0^j})\Big|_{x=\overline{Z}_{t_l}^i}\right| \\
& + C\left(\Delta t + \|V_0 - \overline{V}_0\|_{L^\infty(\mathbb{R})}\right).
\end{aligned}$$

The second term of the right-hand side is a discretization error in the weak sense. It is described by

Lemma 3.11 *For all x and y in $\mathbb{R}$ and all discrete time t_l, $l \in \{0, \ldots, K\}$,*

$$\left|\mathbb{E}H_\varepsilon(x - Z_{t_l}^{0,y}) - \mathbb{E}H_\varepsilon(x - \overline{Z}_{t_l}^{0,y})\right| \leq C\Delta t,$$

where the constant C depend on σ, V_0 and T only and is uniform in x and y.

We admit this result whose proof is postponed at the end of this subsection. The last step consists in introduce a term of statistical nature:

$$\begin{aligned}
\overline{E}_{t_l} \quad \leq \quad & \frac{1}{N^2}\sum_{i,j=1}^{N}\mathbb{E}\left|H_\varepsilon(\overline{Z}_{t_l}^i - \overline{Z}_{t_l}^j) - H_\varepsilon(Y_{t_l}^i - Y_{t_l}^j)\right| \\
& + \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|\frac{1}{N}\sum_{j=1}^{N}\left(\mathbb{E}H_\varepsilon(x - \overline{Z}_{t_l}^{0,y_0^j})\Big|_{x=\overline{Z}_{t_l}^i} - H_\varepsilon(\overline{Z}_{t_l}^i - \overline{Z}_{t_l}^j)\right)\right| \\
& + C\left(\Delta t + \|V_0 - \overline{V}_0\|_{L^\infty(\mathbb{R})}\right).
\end{aligned}$$

Observe that $\overline{Z}_{t_l}^{0,y_0^j}$ and $\overline{Z}_{t_l}^{j}$ have the same law. Let $\mathcal{F}_t^i := \sigma(W_s^i; 0 \leq s \leq t)$. For $j \neq i$, $\overline{Z}_{t_l}^{j}$ and $\overline{Z}_{t_l}^{j}$ being independent, we have

$$\mathbb{E}^{\mathcal{F}_{t_l}^i} \left(\mathbb{E} H_\varepsilon (x - \overline{Z}_{t_l}^{0,y_0^j}) \Big|_{x=\overline{Z}_{t_l}^i} - H_\varepsilon(\overline{Z}_{t_l}^i - \overline{Z}_{t_l}^j) \right) = 0,$$

which implies

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{N} \sum_{j \neq i}^{N} \left(\mathbb{E} H_\varepsilon (x - \overline{Z}_{t_l}^{0,y_0^j}) \Big|_{x=\overline{Z}_{t_l}^i} - H_\varepsilon(\overline{Z}_{t_l}^i - \overline{Z}_{t_l}^j) \right) \right)^2 \\
&= \frac{1}{N^2} \sum_{j \neq i}^{N} \mathbb{E} \left(\mathbb{E} H_\varepsilon (x - \overline{Z}_{t_l}^{0,y_0^j}) \Big|_{x=\overline{Z}_{t_l}^i} - H_\varepsilon(\overline{Z}_{t_l}^i - \overline{Z}_{t_l}^j) \right)^2 \leq \frac{2}{N}.
\end{aligned}$$

Hence, we have obtained that

$$\overline{E}_{t_l} \leq \frac{1}{N^2} \sum_{i,j=1}^{N} \mathbb{E} \left| H_\varepsilon(\overline{Z}_{t_l}^i - \overline{Z}_{t_l}^j) - H_\varepsilon(Y_{t_l}^i - Y_{t_l}^j) \right| + C \left(\Delta t + \|V_0 - \overline{V}_0\|_{L^\infty(\mathbb{R})} + \frac{1}{\sqrt{N}} \right) \tag{3.23}$$

It remains to analyse the first term in the right-hand side. We do so by making appear the successive transitions of the processes $(\overline{Z}^i)$:

$$\begin{aligned}
&\mathbb{E} \left| H_\varepsilon(\overline{Z}_{t_l}^i - \overline{Z}_{t_l}^j) - H_\varepsilon(Y_{t_l}^i - Y_{t_l}^j) \right| \\
&\leq \sum_{n=0}^{l-1} \mathbb{E} \left| H_\varepsilon(\overline{Z}_{t_l}^{i,t_{l-n},Y_{t_{l-n}}^i} - \overline{Z}_{t_l}^{j,t_{l-n},Y_{t_{l-n}}^j}) - H_\varepsilon(\overline{Z}_{t_l}^{i,t_{l-n-1},Y_{t_{l-n-1}}^i} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y_{t_{l-n-1}}^j}) \right|.
\end{aligned}$$

As the drift $B(t,x)$ of $(\overline{Z})$ is a Lipschitz function, one can easily show that, for any i in $\{0, \ldots, N\}$,

$$\begin{aligned}
\left| \overline{Z}_{t_l}^{i,t_{l-n},Y_{t_{l-n}}^i} - \overline{Z}_{t_l}^{i,t_{l-n-1},Y_{t_{l-n-1}}^i} \right| &= \left| \overline{Z}_{t_l}^{i,t_{l-n},Y_{t_{l-n}}^i} - \overline{Z}_{t_l}^{i,t_{l-n},\overline{Z}_{t_{l-n}}^{i,t_{l-n-1},Y_{t_{l-n-1}}^i}} \right| \\
&\leq C \left| Y_{t_{l-n}}^i - \overline{Z}_{t_{l-n}}^{i,t_{l-n-1},Y_{t_{l-n-1}}^i} \right|
\end{aligned}$$

and hence that

$$\left| \overline{Z}_{t_l}^{i,t_{l-n},Y_{t_{l-n}}^i} - \overline{Z}_{t_l}^{i,t_{l-n-1},Y_{t_{l-n-1}}^i} \right| \leq C \Delta t \left| \overline{V}_{t_{l-n-1}}(Y_{t_{l-n-1}}^i) - V(t_{l-n-1}, Y_{t_{l-n-1}}^i) \right|. \tag{3.24}$$

For each term in the sum, we use the identity $H_\varepsilon(a) - H_\varepsilon(b) = (a-b)\int_0^1 g_\varepsilon((a-b)u+b)du$ to get

$$
\begin{aligned}
&\mathbb{E}\left| H_\varepsilon(\overline{Z}_{t_l}^{i,t_{l-n},Y^i_{t_{l-n}}} - \overline{Z}_{t_l}^{j,t_{l-n},Y^j_{t_{l-n}}}) - H_\varepsilon(\overline{Z}_{t_l}^{i,t_{l-n-1},Y^i_{t_{l-n-1}}} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}})\right| \\
&\leq \mathbb{E}\left| H_\varepsilon(\overline{Z}_{t_l}^{i,t_{l-n},Y^i_{t_{l-n}}} - \overline{Z}_{t_l}^{j,t_{l-n},Y^j_{t_{l-n}}}) - H_\varepsilon(\overline{Z}_{t_l}^{i,t_{l-n},Y^i_{t_{l-n}}} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}})\right| \\
&+ \mathbb{E}\left| H_\varepsilon(\overline{Z}_{t_l}^{i,t_{l-n},Y^i_{t_{l-n}}} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}}) - H_\varepsilon(\overline{Z}_{t_l}^{i,t_{l-n-1},Y^i_{t_{l-n-1}}} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}})\right| \\
&\leq \mathbb{E}\Bigg[\left|\overline{Z}_{t_l}^{j,t_{l-n},Y^j_{t_{l-n}}} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}}\right| \\
&\int_0^1 g_\varepsilon\left(u(\overline{Z}_{t_l}^{j,t_{l-n},Y^j_{t_{l-n}}} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}}) + \overline{Z}_{t_l}^{i,t_{l-n},Y^i_{t_{l-n}}} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}}\right) du\Bigg] \\
&+ \mathbb{E}\Bigg[\left|\overline{Z}_{t_l}^{i,t_{l-n},Y^i_{t_{l-n}}} - \overline{Z}_{t_l}^{i,t_{l-n-1},Y^i_{t_{l-n-1}}}\right| \\
&\int_0^1 g_\varepsilon\left(u(\overline{Z}_{t_l}^{i,t_{l-n},Y^i_{t_{l-n}}} - \overline{Z}_{t_l}^{i,t_{l-n-1},Y^i_{t_{l-n-1}}}) + \overline{Z}_{t_l}^{i,t_{l-n-1},Y^i_{t_{l-n-1}}} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}}\right) du\Bigg].
\end{aligned}
$$

Hence, by (3.24), $\left|\overline{V}_{t_{l-n-1}}(Y^j_{t_{l-n-1}}) - V(t_{l-n-1}, Y^j_{t_{l-n-1}})\right|$ being a $\mathcal{F}_{t_{l-n-1}}$-measurable variable,

$$
\begin{aligned}
&\mathbb{E}\left| H_\varepsilon(\overline{Z}_{t_l}^{i,t_{l-n},Y^i_{t_{l-n}}} - \overline{Z}_{t_l}^{j,t_{l-n},Y^j_{t_{l-n}}}) - H_\varepsilon(\overline{Z}_{t_l}^{i,t_{l-n-1},Y^i_{t_{l-n-1}}} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}})\right| \\
&\leq C\Delta t\mathbb{E}\Bigg[\left|\overline{V}_{t_{l-n-1}}(Y^j_{t_{l-n-1}}) - V(t_{l-n-1}, Y^j_{t_{l-n-1}})\right| \\
&\int_0^1 \mathbb{E}^{\mathcal{F}_{t_{l-n-1}}} g_\varepsilon\left(u(\overline{Z}_{t_l}^{j,t_{l-n},Y^j_{t_{l-n}}} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}}) + \overline{Z}_{t_l}^{i,t_{l-n},Y^i_{t_{l-n}}} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}}\right) du\Bigg] \\
&+ C\Delta t\mathbb{E}\Bigg[\left|\overline{V}_{t_{l-n-1}}(Y^i_{t_{l-n-1}}) - V(t_{l-n-1}, Y^i_{t_{l-n-1}})\right| \\
&\int_0^1 \mathbb{E}^{\mathcal{F}_{t_{l-n-1}}} g_\varepsilon\left(u(\overline{Z}_{t_l}^{i,t_{l-n},Y^i_{t_{l-n}}} - \overline{Z}_{t_l}^{i,t_{l-n-1},Y^i_{t_{l-n-1}}}) + \overline{Z}_{t_l}^{i,t_{l-n-1},Y^i_{t_{l-n-1}}} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}}\right) du\Bigg].
\end{aligned}
$$

Now we need to bound

$$
\mathbb{E}^{\mathcal{F}_{t_{l-n-1}}} g_\varepsilon\left(u(\overline{Z}_{t_l}^{j,t_{l-n},Y^j_{t_{l-n}}} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}}) + \overline{Z}_{t_l}^{i,t_{l-n},Y^i_{t_{l-n}}} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}}\right) \quad (3.25)
$$

and

$$
\mathbb{E}^{\mathcal{F}_{t_{l-n-1}}} g_\varepsilon\left(u(\overline{Z}_{t_l}^{i,t_{l-n},Y^i_{t_{l-n}}} - \overline{Z}_{t_l}^{i,t_{l-n-1},Y^i_{t_{l-n-1}}}) + \overline{Z}_{t_l}^{i,t_{l-n-1},Y^i_{t_{l-n-1}}} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}}\right) (3.26)
$$

We only treat the term (3.25), the upper bound of (3.26) can be obtained with similar arguments. First, we remark that the random variables $u(\overline{Z}_{t_l}^{j,t_{l-n},Y^j_{t_{l-n}}} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}}) - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}}$ and $\overline{Z}_{t_l}^{i,t_{l-n},Y^i_{t_{l-n}}}$ are $\mathcal{F}_{t_{l-n-1}}$ -independent. Moreover,

$$
\begin{aligned}
&u(\overline{Z}_{t_l}^{j,t_{l-n},Y^j_{t_{l-n}}} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}}) - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}} \\
&= -Y^j_{t_{l-n-1}} + \int_{t_{l-n-1}}^{t_l} \phi^j_s ds - \sigma(W^j_{t_l} - W^j_{t_{l-n-1}})
\end{aligned}
$$

and

$$
\overline{Z}_{t_l}^{i,t_{l-n},Y^i_{t_{l-n}}} = Y^i_{t_{l-n-1}} + \int_{t_{l-n-1}}^{t_l} \psi^i_s ds - \sigma(W^i_{t_l} - W^i_{t_{l-n-1}}),
$$

where, for all $t \in [t_{l-n-1}, T]$,

$$
\begin{aligned}
\phi^j_t = &\left[u\left(A'(\overline{V}_{t_{l-n-1}}(Y^j_{t_{l-n-1}})) - B(t_{l-n-1}, Y^j_{t_{l-n-1}})\right) - B(t_{l-n-1}, Y^j_{t_{l-n-1}})\right] \mathbb{1}_{[t_{l-n-1}, t_{l-n}[}(t) \\
&+ \sum_{k=l-n}^{K} \left[u\left(B(t_k, \overline{Z}_{t_k}^{j,t_{l-n},Y^j_{t_{l-n}}}) - B(t_k, \overline{Z}_{t_k}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}})\right) - B(t_k, \overline{Z}_{t_k}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}})\right] \mathbb{1}_{[t_k, t_{k+1}[}(t)
\end{aligned}
$$

and

$$
\psi^i_t = A'\left(\overline{V}_{t_{l-n-1}}(Y^i_{t_{l-n-1}})\right) \mathbb{1}_{[t_{l-n-1}, t_{l-n}[}(t) + \sum_{k=l-n}^{K} B\left(t_k, \overline{Z}_{t_k}^{i,t_{l-n},Y^i_{t_{l-n}}}\right) \mathbb{1}_{[t_k, t_{k+1}[}(t).
$$

ϕ^j_t and ψ^i_t are uniformly bounded. Then, by Girsanov's theorem, the laws of $u(\overline{Z}_{t_l}^{j,t_{l-n},Y^j_{t_{l-n}}} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}}) - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}}$ and $\overline{Z}_{t_l}^{i,t_{l-n},Y^i_{t_{l-n}}}$ have densities denoted respectively by $\widetilde{\Gamma}(\cdot, -Y^j_{t_{l-n-1}})$ and $\widehat{\Gamma}(\cdot, Y^i_{t_{l-n-1}})$. Moreover, applying Remark 3.5, $\widetilde{\Gamma}$ and $\widehat{\Gamma}$ are in $L^2(\mathbb{R})$ and

$$
\|\widetilde{\Gamma}(\cdot, -Y^j_{t_{l-n-1}})\|_{L^2(\mathbb{R})} + \|\widehat{\Gamma}(\cdot, Y^i_{t_{l-n-1}})\|_{L^2(\mathbb{R})} \leq \frac{C}{t_{n+1}^{1/4}}.
$$

Hence, the term (3.25) becomes

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} g_\varepsilon(z+y) \widetilde{\Gamma}(y, -Y^j_{t_{l-n-1}}) \widehat{\Gamma}(z, Y^i_{t_{l-n-1}}) dz dy \\
&\leq \|\widetilde{\Gamma}(\cdot, -Y^j_{t_{l-n-1}})\|_{L^2(\mathbb{R})} \|\widehat{\Gamma}(\cdot, Y^i_{t_{l-n-1}})\|_{L^2(\mathbb{R})} \leq \frac{C}{\sqrt{t_{n+1}}}.
\end{aligned}
$$

The same bound holds for (3.26). Thus, combining this estimates with (3.22), for $n \in \{0, \dots, l-1\}$ we get

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\mathbb{E}\left|H_\varepsilon(\overline{Z}_{t_l}^{i,t_{l-n},Y^i_{t_{l-n}}} - \overline{Z}_{t_l}^{j,t_{l-n},Y^j_{t_{l-n}}}) - H_\varepsilon(\overline{Z}_{t_l}^{i,t_{l-n-1},Y^i_{t_{l-n-1}}} - \overline{Z}_{t_l}^{j,t_{l-n-1},Y^j_{t_{l-n-1}}})\right|$$
$$\leq \frac{C}{\sqrt{t_{n+1}}}\left[\overline{E}_{t_{l-n-1}} + C\Delta t\sum_{m=0}^{l-n-2}\overline{E}_{t_m}\right].$$

Summing all the terms for n in $\{0, \dots, l-1\}$, it comes

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\mathbb{E}\left|H_\varepsilon(\overline{Z}^i_{t_l} - \overline{Z}^j_{t_l}) - H_\varepsilon(Y^i_{t_l} - Y^j_{t_l})\right| \leq \sum_{n=1}^{l}\frac{C\Delta t}{\sqrt{t_n}}\overline{E}_{t_{l-n}}.$$

This last bound with (3.23) gives the induction relation

$$\overline{E}_{t_l} \leq \sum_{n=0}^{l-1}\frac{C\Delta t}{\sqrt{t_l - t_n}}\overline{E}_{t_n} + C\left(\Delta t + \|V_0 - \overline{V}_0\|_{L^\infty(\mathbb{R})} + \frac{1}{\sqrt{N}}\right).$$

∎

Proof of Lemma 3.11: To study this weak type error for the Euler scheme, we follow a technique due to Talay and Tubaro [9]. The main idea consists in use the Feynman-Kac representation of Cauchy problem and remark that $\mathbb{E}H_\varepsilon(x - Z^{0,y}_{t_l}) = v_{t_l,x}(0,y)$, where the function $v_{t_l,x}(s,y)$ is solution of the partial differential equation

$$\begin{cases} \dfrac{\partial v_{t_l,x}}{\partial s}(s,y) + \dfrac{1}{2}\sigma^2\dfrac{\partial^2 v_{t_l,x}}{\partial y^2}(s,y) + B(s,y)\dfrac{\partial v_{t_l,x}}{\partial y}(s,y) = 0, \ \forall (s,y) \in [0,t_l) \times \mathbb{R}, \\ v_{t_l,x}(t_l,y) = H_\varepsilon(x-y), \ \forall y \in \mathbb{R}. \end{cases} \tag{3.27}$$

The above Cauchy problem is similar to Equation (3.5) and the results of Lemma 3.9 hold for Equation (3.27), replacing t_k by t_l in the setting. Thus

$$\mathbb{E}H_\varepsilon(x - Z^{0,y}_{t_l}) - \mathbb{E}H_\varepsilon(x - \overline{Z}^{0,y}_{t_l}) = v_{t_l,x}(0,y) - \mathbb{E}v_{t_l,x}(t_l, \overline{Z}^{0,y}_{t_l}).$$

In the sequel, we will use the notation v rather than $v_{t_l,x}$, except when we need to make appear the parameters x and t_l. We decompose the expression above making appear the discrete dates in $[0, t_l)$

$$v_{t_l,x}(0,y) - \mathbb{E}v_{t_l,x}(t_l, \overline{Z}^{0,y}_{t_l}) = \sum_{n=0}^{l-1}\mathbb{E}\left(v(t_{n+1}, \overline{Z}^{0,y}_{t_{n+1}}) - v(t_n, \overline{Z}^{0,y}_{t_n})\right).$$

We apply Itô's formula a first time and use Equation (3.27) to obtain

$$v_{t_l,x}(0,y) - \mathbb{E}v_{t_l,x}(t_l, \overline{Z}_{t_l}^{0,y}) = \sum_{n=0}^{l-1} \mathbb{E}\int_{t_n}^{t_{n+1}} \frac{\partial v}{\partial y}(s, \overline{Z}_s^{0,y}) \left[B(t_n, \overline{Z}_{t_n}^{0,y}) - B(s, \overline{Z}_s^{0,y})\right] ds,$$

where $\overline{Z}_s^{0,y} = \overline{Z}_{t_n}^{0,y} + sB(t_n, \overline{Z}_{t_n}^{0,y}) + \sigma(W_s - W_{t_n})$ when $s \in [t_n, t_{n+1})$. Apply Itô's formula and Equation (3.27) again, it comes

$$\begin{aligned}
&\mathbb{E}\int_{t_n}^{t_{n+1}} \frac{\partial v}{\partial y}(s, \overline{Z}_s^{0,y}) \left[B(t_n, \overline{Z}_{t_n}^{0,y}) - B(s, \overline{Z}_s^{0,y})\right] ds \\
&= \mathbb{E}\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \left(\frac{\partial}{\partial \theta} + B(t_n, \overline{Z}_{t_n}^{0,y})\frac{\partial}{\partial y} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}\right) \left(\frac{\partial v}{\partial y}\left(B(t_n, \overline{Z}_{t_n}^{0,y}) - B\right)\right)(\theta, \overline{Z}_\theta^{0,y}) d\theta ds \\
&= \mathbb{E}\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \left[-\frac{\partial v}{\partial y}\left(\frac{\partial B}{\partial \theta} + B(t_n, \overline{Z}_{t_n}^{0,y})\frac{\partial B}{\partial y} + \frac{\sigma^2}{2}\frac{\partial^2 B}{\partial y^2}\right)\right] (\theta, \overline{Z}_\theta^{0,y}) d\theta ds \\
&\quad + \mathbb{E}\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \left[\frac{\partial^2 v}{\partial y^2}\left((B(t_n, \overline{Z}_{t_n}^{0,y}) - B)^2 - \sigma^2\frac{\partial B}{\partial y}\right)\right] (\theta, \overline{Z}_\theta^{0,y}) d\theta ds.
\end{aligned}$$

Using the bounds on B and its derivatives given in Lemma 3.6, we get

$$\begin{aligned}
&\mathbb{E}H_\varepsilon(x - Z_{t_l}^{0,y}) - \mathbb{E}H_\varepsilon(x - \overline{Z}_{t_l}^{0,y}) \\
&\leq C \sum_{n=0}^{l-1} \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \left(\mathbb{E}\left|\frac{\partial^2 v}{\partial y^2}(\theta, \overline{Z}_\theta^{y,0})\right| + \mathbb{E}\left|\frac{\partial v}{\partial y}(\theta, \overline{Z}_\theta^{y,0})\right|\right) d\theta ds.
\end{aligned}$$

Finally, using the same technique than in the computation of (3.19), we obtain that

$$\mathbb{E}\left|\frac{\partial v_{t_l,x}}{\partial y}(\theta, \overline{Z}_\theta^{y,0})\right| \leq \frac{C}{\sqrt{t_l - \theta}} \quad \text{and} \quad \mathbb{E}\left|\frac{\partial^2 v_{t_l,x}}{\partial y^2}(\theta, \overline{Z}_\theta^{y,0})\right| \leq \frac{C}{\theta^{4/10}(t_l - \theta)^{3/5}},$$

where the constant C is uniform in x and y. We integrate in time to get

$$\left|\mathbb{E}H_\varepsilon(x - Z_{t_l}^{0,y}) - \mathbb{E}H_\varepsilon(x - \overline{Z}_{t_l}^{0,y})\right| \leq C\Delta t.$$

∎

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